JOURNAL OF FUNCTIONAL ANALYSIS **48**, 172–212 (1982)

The Itô–Clifford Integral

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Communicated by L. Gross

Received November 20, 1981

We present a theory of non-commutative stochastic integration analogous to the Itô-theory. It is shown that Wick products of Fermi fields define martingales and that stochastic integrals with respect to these are defined for adapted (operator-valued) square-integrable integrands. For square-integrable martingales associated with an arbitrary probability gage space a stochastic integral is defined, and a Doob–Meyer decomposition for supermartingales obtained.

THE ITÔ–CLIFFORD INTEGRAL

0.1. *Introduction.* This paper constructs an integral of anti-commuting elements analogous to the Itô-integral for Brownian motion. The key notion in the theory of the Itô-integral is the martingale, for which the following set-up is needed: we have a sample space Ω, a complete σ-ring $\mathscr{F}$ of subsets of Ω, and a probability measure μ on $\mathscr{F}$; and we have an increasing family $(\mathscr{F}_t)_{t\geqslant 0}$ of complete σ-subrings of $\mathscr{F}$ (the filtration), which generate $\mathscr{F}$, and obey natural axioms [17, 20]. Then a conditional expectation $M_t = \mathbb{E}(\ |\mathscr{F}_t)$ can be defined, taking an integrable function measurable relative to $\mathscr{F}$ to an integrable function measurable relative to $\mathscr{F}_t$. A martingale is then a process X_t obeying $M_t X_s = X_t$ for all $s \geqslant t$ and all $t \geqslant 0$.

To set up the quantum analogue of this, we note that the classical theory can be reformulated: in place of $(\Omega, \mathscr{F}, \mu)$ we emphasize the Hilbert-space $L^2(\Omega, \mathscr{F}, \mu)$, and in place of the filtration $(\mathscr{F}_t)$ we can consider the family of abelian W^*-algebras $\mathfrak{A}_t = L^\infty(\Omega, \mu, \mathscr{F}_t)_{t\geqslant 0}$ of essentially bounded functions measurable relative to $\mathscr{F}_t$; a set in $\mathscr{F}_t$ can be identified with the multiplication operator by the indicator-function of the set; the conditional expectation is then the projection from $L^2(\Omega, \mu, \mathscr{F})$ onto $L^2(\Omega, \mu, \mathscr{F}_t)$, at least, for square-integrable functions. The expectation of f, $\int f\, d\mu$, is then the expectation value in L^2 in the vector: $\psi(\omega) = 1$ for all ω. A quantum theory of processes would then generalize this by replacing $L^\infty(\Omega, \mu, \mathscr{F}_t)_{0\leqslant t \leqslant \infty}$ by a (non-commutative) W^*-algebra $\mathfrak{A}_t$ of semi-finite type, obeying corresponding

0022-1236/82/110172–41\$02.00/0

axioms. The spaces $L^p(\Omega, \mu, \mathscr{F}_t)$ are then replaced by the "non-commutative" L^p-spaces of Segal [21], and a conditional expectation, a map from $\mathfrak{A}$ to $\mathfrak{A}_t$, can be defined [29]. Martingale theorems and stochastic integrals can then be studied [2]. A brief account of part of this theory is given in Section 7.

0.2. To find examples of quantum martingales, we note that the typical classical martingale, Brownian motion $(B_t)_{t \geq 0}$, has a realization in terms of the Fock representation of the quantized boson field over $L^2(0, \infty) = \mathscr{H}$. Namely, $B_t = \int_0^t \phi(s)\, ds = \phi(\chi_{[0,t]})$, where $\phi(f)$, $f \in \mathscr{H}$ is the Fock field smeared with f. In these terms, the nth order Wiener chaos, $H_n(B_t)$, is the Wick-ordered power [28]

$$H_n(B_t) = :\phi(\chi_{[0,t]})^n:.$$

Further examples of martingales are given by Wick ordering in continuous tensor products [14, 25].

An anti-commuting analogue of Brownian motion is obtained by replacing the Boson field by the Fermion field, $\psi(t)$, and the Brownian motion is replaced by $\Psi(t) = \int_0^t \psi(s)\, ds = \psi(\chi_{[0,t]})$. This is described in detail in Section 1: the construction leads to the Clifford W^*-algebra, $\mathscr{C}$, the hyperfinite II_1 factor, and a family $(\mathscr{C}_t)$ of type II_1 subfactors giving the filtration. In Sections 1–6, the notation $\mathscr{C}$ for $\mathfrak{A}$ and $\mathscr{C}_t$ for $\mathfrak{A}_t$ is used to denote this special case. In Section 2, the conditional expectation is given explicitly and we show that Wick polynomials in smeared fields define martingales.

In Section 3, stochastic integrals of L^2-adapted processes with respect to simple martingales are defined. As in the Itô case, there is an isometry between the space of stochastic integrals and a Hilbert-space of processes. The stochastic integral defines a centred L^2-martingale.

We end the section by a theorem of the Doob–Meyer type, namely, that the submartingale $(\int_0^t f(s)\, d\Psi(s))^* (\int_0^t f(s)\, d\Psi(s))$ can be written as the sum of a martingale and a positive increasing process.

In Section 4 we show that every centred L^2-martingale X_t is a stochastic integral, $X_t = \int_0^t f(s)\, d\Psi(s)$, with f an essentially unique adapted process. We can therefore define the stochastic derivative $\partial X_t / \partial \Psi(t)$ to be $f(t)$. The method exploits Segal's duality map between $L^2(\mathscr{C})$ and Fock space.

In Section 5 a similar theory of stochastic integration is developed for integrals relative to Wick monomials of degree $n > 1$; we use similar methods to the case where $n = 1$.

In Section 6 we show that the stochastic integral converges (as a limit of step-function approximations) in each L^p provided that the integrand f is of finite degree. This is a simple consequence of the hypercontractive estimates for fermions [11, 31]. In Section 7, we give a survey of part of [2]. A form of Doob–Meyer decomposition is proved, and this is used to construct stochastic integrals with respect to a martingale whose square is of "class D".

0.3. The concept of quantum stochastic process used here is looser than that adopted in some recent work [1]. We do not require that the structure of the algebra generated by the process at time t should be independent of t. For example, the algebraic relation $\Psi(t)^2 = t$, true in our theory, depends on time. This appears inevitable if our concept of "quantum stochastic process" is to include some martingales. In applications to atomic systems we should regard $\Psi(t)$ as the "noise" driving an equation of motion; it is the solution to the equation that is the process of direct physical interest. One such model is described in [27].

1. The Fermion Gage Space [24]

1.1. Let $\mathscr{H}$ be a complex Hilbert-space (the fermion "one-particle space"), and let J be a conjugation on $\mathscr{H}$. The anti-symmetric Fock space over $\mathscr{H}$ is the Hilbert-space $\Lambda(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \Lambda_n(\mathscr{H})$, where $\Lambda_0(\mathscr{H}) = \mathbb{C}$ and $\Lambda_n(\mathscr{H})$ is the Hilbert-space anti-symmetric n-fold tensor product of $\mathscr{H}$ with itself.

For each $z \in \mathscr{H}$, the creation operator $C(z)$ is defined by $C(z)\colon \Lambda_n(\mathscr{H}) \to \Lambda_{n+1}(\mathscr{H})$, $u \mapsto (n+1)^{1/2}\ \mathscr{A}(z \otimes u)$, where $\mathscr{A}$ is the anti-symmetrization projection. By linearity and continuity, $C(z)$ defines a bounded operator on $\Lambda(\mathscr{H})$ with norm $\|C(z)\| = \|z\|$ [5].

The annihilation operator $A(z)$ is the adjoint of $C(z)$; $A(z) = C(z)^*$. The fermion field $\Psi(z)$ is defined on $\Lambda(\mathscr{H})$ by $\Psi(z) = C(z) + A(Jz)$. Evidently, $\Psi(\cdot)\colon \mathscr{H} \to \mathscr{B}(\Lambda(\mathscr{H}))$ is linear, and the anti-commutation relations hold;

$$\{\Psi(z), \Psi(z')\} \equiv \Psi(z)\,\Psi(z') + \Psi(z')\,\Psi(z) = 2(Jz', z)\,\mathbb{1}$$

for z, z' in $\mathscr{H}$.

Furthermore, if z is J-real ($Jz = z$) then $\Psi(z)$ is self-adjoint.

Let $\mathscr{C}$ denote the W^*-algebra generated by the bounded operators $\{\Psi(z)\colon z \in \mathscr{H}\}$. $\mathscr{C}$ is called the weakly-closed Clifford operator-algebra of $(\mathscr{H}, J)$.

The Fock vacuum (or "no-particle vector") is the vector $\Omega = 1 \in \Lambda_0(\mathscr{H}) \subset \Lambda(\mathscr{H})$. It is well-known that Ω is cyclic for $\mathscr{C}$ and that $m(\cdot) = (\Omega, \cdot\Omega)$ is a faithful, central state on $\mathscr{C}$, and so $(\Lambda(\mathscr{H}), \mathscr{C}, m)$ is a regular probability gage space.

If $\dim \mathscr{H} = \infty$, $\mathscr{C}$ is the uniformly hyperfinite II_1 factor.

For $1 \leqslant p < \infty$, $L_p(\mathscr{C})$ is the completion of $\mathscr{C}$ with respect to the norm $\|u\|_p = m(|u|^p)^{1/p} = (\Omega, (u^*u)^{p/2}\,\Omega)^{1/p}$. The elements of $L^p(\mathscr{C})$ can be identified with (possibly unbounded) operators on $\Lambda(\mathscr{H})$ [21].

$L^\infty(\mathscr{C})$ is, by definition, $\mathscr{C}$ equipped with its operator norm.

The map $u \mapsto u\Omega$ from $\mathscr{C}$ into $\Lambda(\mathscr{H})$ extends to a unitary operator $D: L^2(\mathscr{C}) \to \Lambda(\mathscr{H})$, and is called the duality transform [24]. Under D, the action of $\mathscr{C}$ on $\Lambda(\mathscr{H})$ becomes left multiplication on $L^2(\mathscr{C})$ (so that $(L^2(\mathscr{C}), \mathscr{C}, m)$ is standard [21]).

Many of the conventional L^p-theory results are also valid for these non-commutative L^p-spaces. In particular, $L^p(\mathscr{C}) \supseteq L^q(\mathscr{C})$ and $\|\cdot\|_p \leqslant \|\cdot\|_q$ for $1 \leqslant p \leqslant q \leqslant \infty$, and $L^{p'}(\mathscr{C})$ is the dual of $L^p(\mathscr{C})$ for $1 \leqslant p < \infty$, where $p' = p/(p-1)$ [6, 16].

If $\mathscr{B}$ is a W^*-subalgebra of $\mathscr{C}$, then $L^p(\mathscr{B})$ is the completion of $\mathscr{B}$ with respect to $\|\cdot\|_p$ and so can be considered as a closed subspace of $L^p(\mathscr{C})$.

1.2 Theorem [30]. *For any $1 \leqslant p \leqslant \infty$, there is a unique map $L^p(\mathscr{C}) \to L^p(\mathscr{B})$, $u \mapsto \hat{u}$ satisfying*

$$m(\hat{u}v) = m(uv)$$

for all $u \in L^p(\mathscr{C})$, $v \in L^{p'}(\mathscr{B})$, with $p' = p/(p-1)$.

$\hat{u}$ is called the conditional expectation of u with respect to $\mathscr{B}$, and is denoted by $m(u|\mathscr{B})$. $m(\cdot|\mathscr{B})$ enjoys the following usual properties [11, 30]:

(i) $m(\cdot|\mathscr{B})$ is a contraction from $L^p(\mathscr{C})$ onto $L^p(\mathscr{B})$ for all $1 \leqslant p \leqslant \infty$;

(ii) $m(\cdot|\mathscr{B})$ is positivity preserving;

(iii) $m(vu|\mathscr{B}) = vm(u|\mathscr{B})$ for all $u \in L^p(\mathscr{C})$, $v \in L^{p'}(\mathscr{B})$;

(iv) if $\mathscr{B}_1 \subseteq \mathscr{B}_2$, then $m(m(\cdot|\mathscr{B}_2)|\mathscr{B}_1) = m(\cdot|\mathscr{B}_1)$.

$m(\cdot|\mathscr{B})$ on $L^2(\mathscr{C})$ is the projection onto the subspace $L^2(\mathscr{B})$.

(We note that the L^p-spaces, conditional expectations, etc., can be constructed from any probability gage space and do not depend on any special properties of the fermion field [6, 11, 16, 21, 22, 26, 29, 32].)

2. Quantum Martingales

Let $\mathscr{H} = L^2(\mathbb{R}_+, ds)$ and let J be the complex conjugation on $L^2(\mathbb{R}_+, ds)$. For given $0 \leqslant t < \infty$, define $\mathscr{C}_t$ to be the W^*-subalgebra of $\mathscr{C}$ generated by the fields $\Psi(u)$ for $u \in L^2(\mathbb{R}_+, ds)$ with ess supp $u \subseteq [0, t]$. Clearly $\mathscr{C}_s \subseteq \mathscr{C}_t$ for $0 \leqslant s \leqslant t$, and $\mathscr{C}$ is generated by the $\mathscr{C}_t$, $t \in \mathbb{R}_+$. We shall denote $m(\cdot|\mathscr{C}_s)$ by $M_s(\cdot)$, $s \in \mathbb{R}_+$.

2.1. Definition. An L^p-martingale adapted to the family $\{\mathscr{C}_t: t \in \mathbb{R}_+\}$ is a collection $\{X_t: t \in \mathbb{R}_+\}$ of operators on $\Lambda(\mathscr{H})$ such that $X_t \in L^p(\mathscr{C}_t)$ for $t \in \mathbb{R}_+$, and $M_s(X_t) = X_s$ for any $0 \leqslant s \leqslant t$.

Clearly, if $X \in L^p(\mathscr{C})$, then $\{M_t(X): t \in \mathbb{R}_+\}$ is an L^p-martingale. In fact, for $1 < p < \infty$, it can be shown that an L^p-martingale X_t is of this form if and only if $\sup_t \|X_t\|_p < \infty$ (for $p = 1$, $\{X_t\}$ is required to be weakly relatively compact [2, 3]).

Let $u_1, ..., u_n \in L^2_{\text{loc}}(\mathbb{R}_+, ds)$; so that for any $t \geqslant 0$, $u_i \chi_{[0,t]} \in L^2(\mathbb{R}_+, ds)$, $1 \leqslant i \leqslant n$. Denote by $W(u_1, ..., u_n; t)$ the Wick-ordered product $:\Psi(u_1 \chi_{[0,t]}) \cdots \Psi(u_n \chi_{[0,t]}):$. For $n = 1$, $W(u; t) = \Psi(u \chi_{[0,t]})$.

In general, a polynomial function of martingales is not a martingale. However, it is well-known [13] that Hermite polynomials of a Wiener process are martingales, and that Hermite polynomials in a free boson field are just the Wick powers of the field [10]. We shall see that this also holds for the fermion field; i.e., $\{W(u_1, ..., u_n; t): t \in \mathbb{R}_+\}$ is an L^∞-martingale adapted to the family $\{\mathscr{C}_t: t \in \mathbb{R}_+\}$. (For bosons, the Wick powers belong to $L^p(Q)$ for all $1 \leqslant p < \infty$, but not to $L^\infty(Q)$, where Q is the appropriate probability space.)

We shall first consider the conditional expectation map. If $S \in \mathscr{B}(\mathscr{H})$ with $\|S\| \leqslant 1$, there is a bounded operator $\Gamma(S)$ on $\Lambda(\mathscr{H})$ whose action on $\Lambda_n(\mathscr{H})$ is given by $S \otimes \cdots \otimes S$ (n factors) for $n \geqslant 1$, and is the identity on $\Lambda_0(\mathscr{H}) = \mathbb{C}$ [23]. If e_t denotes the projection on $\mathscr{H}$ given by $e_t u = \chi_{[0,t]} u$, then it is easy to see that $D^{-1}\Gamma(e_t) D$ is equal to the conditional expectation map $M_t(\cdot)$ on $L^2(\mathscr{C})$. This follows from the easily established facts: $\Gamma(e_t) \Lambda(\mathscr{H}) = \Lambda(e_t \mathscr{H})$, $DL^2(\mathscr{C}) = \Lambda(\mathscr{H})$, $DL^2(\mathscr{C}_t) = \Lambda(e_t \mathscr{H})$, and $M_t(\cdot)$ is the projection of $L^2(\mathscr{C})$ onto $L^2(\mathscr{C}_t)$.

2.2 THEOREM. *$\{W(u_1, ..., u_n; t): t \in \mathbb{R}_+\}$ is an L^∞-martingale adapted to the family $\{\mathscr{C}_t: t \in \mathbb{R}_+\}$.*

Proof. First we note that $W(u_1, ..., u_n; t) \in L^\infty(\mathscr{C}_t) = \mathscr{C}_t$. This is true because $W(u_1, ..., u_n; t)$ can be expressed as a polynomial in the fields $\Psi(u_i \chi_{[0,t]})$, $1 \leqslant i \leqslant n$. Now let $0 \leqslant s \leqslant t$. We have

$$W(u_1, ..., u_n; t) = D^{-1} W(u_1, ..., u_n; t)\, \Omega$$

and so

$$\begin{aligned} M_s(W(u_1, ..., u_n; t)) &= D^{-1}\Gamma(e_s)\, DD^{-1} W(u_1, ..., u_n; t)\, \Omega \\ &= D^{-1}\Gamma(e_s)\, C(u_1 \chi_{[0,t]}) \cdots C(u_n \chi_{[0,t]})\, \Omega \\ &= D^{-1} C(u_1 \chi_{[0,s]}) \cdots C(u_n \chi_{[0,s]})\, \Omega \\ &= D^{-1} W(u_1, ..., u_n; s)\, \Omega \\ &= W(u_1, ..., u_n; s). \end{aligned}$$

Q.E.D.

We shall find further examples of martingales later on. Let us note that if, say, $u_1 \notin L^2(\mathbb{R}_+, ds)$ (and, of course, none of $u_2,\dots, u_n$ are the zero function) then $\sup_t \| W(u_1,\dots, u_n; t)\|_\infty = \infty$ and so $W(u_1,\dots, u_n; t)$ is not of the form $M_t(X)$ for any $X \in L^\infty(\mathscr{C})$. Indeed, in general, $W(u_1,\dots, u_n; t)$ is not of the form $M_t(X)$ for any $X \in L^1(\mathscr{C})$.

We also note that because of the linearity and continuity of $M_t(\cdot)$, linear combinations and L^p-limits of L^p-martingales are also L^p-martingales.

The fermion field provides a non-commutative counterexample to Lévy's theorem that if (X_t) and $(X_t^2 - t)$ are both martingales, then X_t is a Wiener process. Indeed, if we set $X_t = \Psi(\chi_{[0,t]})$, then (X_t) is a martingale, and $X_t^2 - t = 0$ for all t, so $(X_t^2 - t)$ is also a martingale. Clearly (X_t) is not a Wiener process.

Let $\mathscr{W}$ denote the complex linear space of finite linear combinations of Wick martingales and $\mathbb{1}$.

2.3 THEOREM. *Let $\{X_t: t \in \mathbb{R}_+\}$ be an L^2-martingale adapted to the family $\{\mathscr{C}_t: t \in \mathbb{R}_+\}$. Then, for any $T > 0$, there is a sequence (Y_n) in $\mathscr{W}$ such that $Y_n(t) \to X_t$ in $L^2(\mathscr{C})$, uniformly for $t \in [0, T]$.*

Proof. Vectors of the form $:\Psi(v_1) \cdots \Psi(v_n): \Omega$, $n = 1, 2,\dots, v_1,\dots, v_n \in \mathscr{H}$ and Ω are total in $\Lambda(\mathscr{H})$. It follows that if $\mathfrak{W}$ denotes the linear span of Wick monomials and $\mathbb{1}$, then $\mathfrak{W}$ is dense in $L^2(\mathscr{C})$. Hence there is a sequence (W_n) in $\mathfrak{W}$ such that $W_n \to X_T$ in $L^2(\mathscr{C})$.

For $0 \leqslant t \leqslant T$, we have

$$
\begin{aligned}
\|M_t(W_n) - X_t\|_2 &= \|M_t(W_n - X_T)\|_2 \\
&\leqslant \| W_n - X_T\|_2.
\end{aligned}
$$

The result follows since $M_t(W_n) \in \mathscr{W}$. Q.E.D.

L^2-martingales can be simply characterized with the aid of Wick products of the "field at a point," $\psi(x)$. Indeed, let $\psi(x)$ denote the fermion field $\psi(u)$, where u has been replaced formally by a delta-function at x, $x \in \mathbb{R}_+$. Then $\psi(x)$ and Wick products $:\psi(x_1) \cdots \psi(x_n):$ are well-defined as operator-valued distributions [10, 12]. If $w_n \in \Lambda_n(\mathscr{H})$, then $\int w_n(x_1,\dots, x_n): \psi(x_1) \cdots \psi(x_n): dx_1 \cdots dx_n$ symbolically denotes the element in $L^2(\mathscr{C})$ such that

$$
\int w_n(x_1,\dots, x_n) :\psi(x_1) \cdots \psi(x_n): dx_1 \cdots dx_n \Omega = \sqrt{n!}\, w_n.
$$

2.4 THEOREM. *Let $\{X_t: t \in \mathbb{R}_+\}$ be an L^2-martingale adapted to the* *family* $\{\mathscr{C}_t : t \in \mathbb{R}_+\}$. *Then there is* $w_0 \in \mathbb{C}$ *and anti-symmetric functions* $w_n \in L^2_{\text{loc}}(\mathbb{R}^n_+)$, $n = 1, 2, \ldots$, *such that*

$$X_t = w_0 \mathbb{1} + \sum_{n=1}^{\infty} \int w_n(x_1, \ldots, x_n)\, \chi_{[0,t]^n}(x_1, \ldots, x_n) \times :\psi(x_1) \cdots \psi(x_n): dx_1 \cdots dx_n.$$

Proof. Write $X_t = \sum_{n=0}^{\infty} X_t^{(n)}$ where $X_t^{(n)} \Omega \in \Lambda_n(\mathscr{H})$. Then clearly $X_t^{(0)} = w_0 \mathbb{1}\ \forall t$ for some $w_0 \in \mathbb{C}$, and $M_s(X_t)\, \Omega = \Gamma(e_s)\, X_t \Omega$ implies that $X_t^{(n)} \Omega$ has the form $X_t^{(n)} \Omega = \sqrt{n!}\, w_n \chi_{[0,t]^n}$ form some anti-symmetric function $w_n \in L^2_{\text{loc}}(\mathbb{R}^n_+)$. Q.E.D.

3. The Itô–Clifford Stochastic Integral

We shall show, in this section, that one can define a stochastic integral with respect to the fermion field just as one constructs the Itô-integral with respect to a Wiener process. In particular, the stochastic integral defines centred martingales, and an L^2-isometry property also holds.

3.1 Definition. Let $0 \leqslant t_0 \leqslant t$. An adapted process on $[t_0, t]$ is a map $f\colon [t_0, t] \to L^1(\mathscr{C})$ such that $f(s) \in L^1(\mathscr{C}_s)$ for all $s \in [t_0, t]$. Since we shall deal exclusively with adapted processes, we shall henceforth drop the adjective. An L^p-process is a process f such that $f(s) \in L^p(\mathscr{C}_s)$ for all s.

3.2 Definition. A process h on $[t_0, t]$ is said to be simple if it can be expressed as

$$h = \sum_{k=1}^{n} h_{k-1} \chi_{[t_{k-1}, t_k)}$$

on $[t_0, t)$, for some $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t$ and $h_k \in L^1(\mathscr{C})$, $1 \leqslant k \leqslant n-1$.

Since h is a process, we see that $h(s) = h_{k-1} \in L^1(\mathscr{C}_s)$ for all $t_{k-1} \leqslant s < t_k$, i.e., $h_{k-1} \in L^1(\mathscr{C}_{t_{k-1}})$.

Let $u \in L^2_{\text{loc}}(\mathbb{R}_+, ds)$ be real-valued. We shall denote $\Psi(u\chi_{[0,t]})$ by Ψ_t. Then Ψ_t is self-adjoint and belongs to $L^\infty(\mathscr{C}_t)$ for all $t \in \mathbb{R}_+$. Furthermore, we have, for $0 \leqslant s \leqslant t$.

$$(\Psi_t - \Psi_s)^2 = \Psi(u\chi_{[s,t]})^2 = \int_s^t |u(x)|^2\, dx.$$

For each t, Ψ_t is a (version of the) Bernoulli distribution on the two-point set $\{-\alpha_t, \alpha_t\}$, where $\alpha_t^2 = \int_0^t |u(x)|^2\, dx$, and so is centred, i.e., $m(\Psi_t) = 0$ [24].

Furthermore, Ψ_t has independent increments; if $t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4$, then for any bounded Baire functions ϕ, λ on $\mathbb{R}$, we have

$$m(\phi(\Psi_{t_2} - \Psi_{t_1})\,\lambda(\Psi_{t_4} - \Psi_{t_3})) = m(\phi(\Psi_{t_2} - \Psi_{t_1}))\,m(\lambda(\Psi_{t_4} - \Psi_{t_3})).$$

This is because the W^*-algebras generated by $\{\Psi(z): z \in K_1\}$ and $\{\Psi(z'): z' \in K_2\}$ are independent whenever K_1 and K_2 are orthogonal subspaces of $\mathscr{H}$ [24].

3.3 Definition. If $h = \sum h_{k-1}\chi_{[t_{k-1}, t_k)}$ is a simple process on $[t_0, t]$, the Itô–Clifford stochastic integral of h over $[t_0, t]$ is

$$\int_{t_0}^{t} h(s)\,d\Psi_s = \sum_{k=1}^{n} h_{k-1}(\Psi_{t_k} - \Psi_{t_{k-1}}).$$

Clearly $\int_{t_0}^{t} h(s)\,d\Psi_s \in L^1(\mathscr{C}_t)$, and is independent of the decomposition of h as a sum of step-functions. For notational convenience we will sometimes write $I(h)$ for $\int_{t_0}^{t} h(s)\,d\Psi_s$ and $\Delta\Psi_k$ for $\Psi_{t_k} - \Psi_{t_{k-1}}$.

3.4 Lemma. *Let $\{X_t: t \in \mathbb{R}_+\}$ be an L^p-martingale, and suppose that $0 \leqslant s \leqslant t$. Then $m(f(X_t - X_s)\,g) = 0$ for any $f \in L^q(\mathscr{C}_s)$, $g \in L^r(\mathscr{C}_s)$ with $1/p + 1/q + 1/r = 1$. In particular, $m(f(\Psi_t - \Psi_s)\,g) = 0$ for any $f \in L^q(\mathscr{C}_s)$, $g \in L^r(\mathscr{C}_s)$ with $1/q + 1/r = 1$.*

Proof. We have

$$\begin{aligned} m(f(X_t - X_s)\,g) &= m(gf(X_t - X_s)) \\ &= m(gfM_s(X_t - X_s)) \\ &= 0. \end{aligned}$$

The last part follows since $\Psi_t \in L^\infty$ for all t. Q.E.D.

3.5 Theorem. *The Itô–Clifford stochastic integral satisfies the following:*

(a) $\int_{t_0}^{t} (\alpha h(s) + \beta g(s))\,d\Psi_s = \alpha \int_{t_0}^{t} h(s)\,d\Psi_s + \beta \int_{t_0}^{t} g(s)\,d\Psi_s$ *for simple processes h, g and $\alpha, \beta \in \mathbb{C}$.*

(b) $m(\int_{t_0}^{t} h(s)\,d\Psi_s) = 0$, *for simple h.*

(c) *If h is a simple L^2-process on $[t_0, t]$, i.e., $h(s) \in L^2(\mathscr{C}_s)$ for all $t_0 \leqslant s \leqslant t$, then $\int_{t_0}^{t} h(s)\,d\Psi_s \in L^2(\mathscr{C}_t)$ and $\|\int_{t_0}^{t} h(s)\,d\Psi_s\|_2^2 = \int_{t_0}^{t} \|h(s)\|_2^2\,|u(s)|^2\,ds$.*

Proof. (a) This is clear.

(b) We have

$$m\left(\int_{t_0}^{t} h d\Psi\right) = \sum_{k=1}^{n} m(h_{k-1}(\Psi_{t_k} - \Psi_{t_{k-1}}))$$

for suitable $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t$ and $h_{k-1} \in L^1(\mathscr{C}_{t_{k-1}})$,

$$= 0 \qquad \text{by Lemma 3.4.}$$

(c)

$$\left\| \int_{t_0}^{t} h d\Psi \right\|_2^2 = m\left(\left\{\sum_{k=1}^{n} h_{k-1} \Delta\Psi_k\right\}^* \left\{\sum_{j=1}^{n} h_{j-1} \Delta\Psi_j\right\}\right)$$
$$= \sum_{k,j} m(\Delta\Psi_k h_{k-1}^* h_{j-1} \Delta\Psi_j).$$

By Lemma 3.4 we see that the off-diagonal (i.e., $k \neq j$) terms all vanish, so we need only consider the terms with $k = j$.

However,

$$m(\Delta\Psi_k h_{k-1}^* h_{k-1} \Delta\Psi_k) = m(h_{k-1}^* h_{k-1} (\Delta\Psi_k)^2)$$
$$= m(h_{k-1}^* h_{k-1}) \int_{t_{k-1}}^{t_k} |u(s)|^2 \, ds$$
$$= \int_{t_0}^{t} m(h_{k-1}^* h_{k-1}) \chi_{[t_{k-1}, t_k]}(s) \, |u(s)|^2 \, ds.$$

Summing over k gives the result. Q.E.D.

We shall call property (c) the isometry property of the Itô–Clifford stochastic integral. It is this property which will allow us to define the stochastic integral for processes other than simple ones. First we need some preliminary results.

Let $d\mu$ denote the measure $|u(s)|^2 \, ds$ on $\mathbb{R}_+$.

3.6 THEOREM. *Let g be a continuous L^2-process on $[t_0, t]$; i.e., $s \mapsto g(s)$ is continuous from $[t_0, t]$ into $L^2(\mathscr{C})$. Then, for given $\varepsilon > 0$, there is a simple L^2-process h on $[t_0, t]$ such that*

$$\int_{t_0}^{t} \| g(s) - h(s) \|_2^2 \, d\mu(s) < \varepsilon.$$

Proof. Let $\varepsilon' > 0$ be given. Since g is continuous, it is uniformly continuous on $[t_0, t]$, and so there is $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t$ such that

$$\| g(s) - g(t_{k-1})\|_2^2 < \varepsilon'$$

whenever $t_{k-1} \leqslant s \leqslant t_k$.

Putting $h = \sum_{k=1}^n g(t_{k-1}) \chi_{[t_{k-1}, t_k)}$, we see that $\| g(s) - h(s)\|_2^2 < \varepsilon'$ for all $t_0 \leqslant s < t$, and so

$$\int_{t_0}^t \| g(s) - h(s)\|_2^2 \, d\mu(s) < \varepsilon' \int_{t_0}^t d\mu(s). \qquad \text{Q.E.D.}$$

3.7 LEMMA. *For fixed $X \in L^2(\mathscr{C})$, the map $s \mapsto M_s(X)$ is continuous from $\mathbb{R}_+$ into $L^2(\mathscr{C})$.*

Proof. The map $s \mapsto e_s$ is strongly continuous on $\mathscr{H} = L^2(\mathbb{R}_+, dx)$ and so $\Gamma(e_s)$ is strongly continuous on $\Lambda(\mathscr{H})$. Therefore $s \mapsto M_s(X) = D^{-1}\Gamma(e_s) DX$ is continuous from $\mathbb{R}_+$ into $D^{-1}\Lambda(\mathscr{H}) = L^2(\mathscr{C})$. Q.E.D.

This result also follows from the L^p-martingale convergence theorem [2, 3].

We recall that $L^2([t_0, t], d\mu; L^2(\mathscr{C}))$, the complex Hilbert-space of $L^2(\mathscr{C})$-valued measurable maps on $[t_0, t]$, square-integrable with respect to $d\mu$, is isomorphic to the Hilbert-space tensor product $L^2([t_0, t], d\mu) \otimes L^2(\mathscr{C})$. This, in turn, is the completion of the algebraic tensor product $C([t_0, t]) \otimes L^2(\mathscr{C})$, where $C([t_0, t])$ is the space of complex-valued continuous functions on $[t_0, t]$. (The element $\phi(\cdot) \otimes X$ is identified with $\phi(\cdot) X$ in the above isomorphism.)

We will consider elements of $L^2([t_0, t], d\mu; L^2(\mathscr{C}))$ as maps: $[t_0, t] \to L^2(\mathscr{C})$ defined μ almost everywhere rather than equivalence classes of maps. Thus, $f \in L^2([t_0, t], d\mu; L^2(\mathscr{C}))$ is a process if $f(s) \in L^2(\mathscr{C}_s)$ μ a.e. Let $\mathfrak{H}[t_0, t]$ denote the set of processes in $L^2([t_0, t]\, d\mu; L^2(\mathscr{C}))$.

3.8 PROPOSITION. *$\mathfrak{H}[t_0, t]$ is a closed subspace of $L^2([t_0, t]\, d\mu; L^2(\mathscr{C}))$; i.e., $\mathfrak{H}[t_0, t]$ is a Hilbert-space.*

Proof. Let (f_n) be a sequence in $\mathfrak{H}[t_0, t]$ such that $f_n \to f$ in $L^2([t_0, t], d\mu; L^2(\mathscr{C}))$. By passing to a subsequence, we may suppose that $f_n(s) \to f(s)$ μ a.e. in $L^2(\mathscr{C})$. But $f_n(s) \in L^2(\mathscr{C}_s)$ μ a.e. and so $f(s) \in L^2(\mathscr{C}_s)$ μ a.e. Hence $f \in \mathfrak{H}[t_0, t]$. Q.E.D.

3.9 THEOREM. *Let $f \in \mathfrak{H}[t_0, t]$. Then f can be approximated arbitrarily closely in $\mathfrak{H}[t_0, t]$ by simple processes in $\mathfrak{H}[t_0, t]$.*

Proof. For given $\varepsilon > 0$ there is $n \in \mathbb{N}$ and $\phi_j \in C([t_0, t])$, $X_j \in L^2(\mathscr{C})$, $1 \leqslant j \leqslant n$ such that

$$\int_{t_0}^{t} \left\| f(s) - \sum_{j=1}^{n} \phi_j(s) X_j \right\|_2^2 d\mu(s) < \varepsilon^2.$$

Since f is a process and $M_s \colon L^2(\mathscr{C}) \to L^2(\mathscr{C}_s)$ is a contraction, we have

$$\begin{aligned}\int_{t_0}^{t} & \left\| f(s) - \sum_{j=1}^{n} \phi_j(s) M_s(X_j) \right\|_2^2 d\mu(s) \\ &= \int_{t_0}^{t} \left\| M_s \left(f(s) - \sum_{j=1}^{n} \phi_j(s) X_j \right) \right\|_2^2 d\mu(s) \\ &\leqslant \varepsilon^2.\end{aligned}$$

By Lemma 3.7, each $\phi_j(\cdot) M \cdot (X_j)$ is a continuous L^2-process (and so belongs to $\mathfrak{H}[t_0, t]$) and can therefore be approximated by simple processes in $\mathfrak{H}[t_0, t]$, by Theorem 3.6. The result now follows since a finite linear combination of simple processes is a simple process. Q.E.D.

3.10 Theorem. *For $f \in \mathfrak{H}[t_0, t]$, there is a sequence (h_n) of simple processes converging to f in $\mathfrak{H}[t_0, t]$, and there is $I(f) \in L^2(\mathscr{C}_t)$ such that $\int_{t_0}^{t} h_n(s)\, d\Psi_s$ converges to $I(f)$ in $L^2(\mathscr{C}_t)$. Moreover, $I(f)$ is independent of the particular sequence (h_n) converging to f.*

Proof. The existence of (h_n) converging to f in $\mathfrak{H}[t_0, t]$ is guaranteed by Theorem 3.9. In particular, (h_n) is a Cauchy sequence in $\mathfrak{H}[t_0, t]$, and therefore, by the isometry property, $(\int_{t_0}^{t} h_n\, d\Psi)$ is a Cauchy sequence in $L^2(\mathscr{C}_t)$. The existence of the required $I(f)$ follows from the completeness of $L^2(\mathscr{C}_t)$.

It is easy to see that $I(f)$ is independent of the particular sequence (h_n). Q.E.D.

3.11 Definition. For $f \in \mathfrak{H}[t_0, t]$, the Itô–Clifford stochastic integral of f is $\int_{t_0}^{t} f(s)\, d\Psi_s = I(f)$, where $I(f) \in L^2(\mathscr{C}_t)$ is as given by Theorem 3.10.

If $0 \leqslant t_0 \leqslant t_1 \leqslant t_2$ and $f \in \mathfrak{H}[t_0, t_2]$, then the restrictions of f to $[t_0, t_1]$ and $[t_1, t_2]$ belong to $\mathfrak{H}[t_0, t_1]$ and $\mathfrak{H}[t_1, t_2]$, respectively, and

$$\int_{t_0}^{t_2} f\, d\Psi_s = \int_{t_0}^{t_1} f\, d\Psi_s + \int_{t_1}^{t_2} f\, d\Psi_s.$$

3.12 THEOREM. (a) $\int_{t_0}^t (\alpha f + \beta g)\, d\Psi_s = \alpha \int_{t_0}^t f\, d\Psi_s + \beta \int_{t_0}^t g\, d\Psi_s$, *for any* f, g *in* $\mathfrak{H}[t_0, t]$, $\alpha, \beta \in \mathbb{C}$.

(b) $m(\int_{t_0}^t f\, d\Psi_s) = 0$, *for any* $f \in \mathfrak{H}[t_0, t]$.

(c) $\|\int_{t_0}^t f\, d\Psi_s\|_2 = \|f\|_{\mathfrak{H}}$, *for any* $f \in \mathfrak{H}[t_0, t]$.

(d) $M_{t_0}(\int_{t_0}^t f\, d\Psi_s) = 0$, *for any* $f \in \mathfrak{H}[t_0, t]$.

(e) *If* $f \in \mathfrak{H}[0, t]$ *for all* $t \in \mathbb{R}_+$, *then* $\{\int_0^t f\, d\Psi_s : t \in \mathbb{R}_+\}$ *is an* L^2-*martingale adapted to* $\{\mathscr{C}_t : t \in \mathbb{R}_+\}$.

Proof. Parts (a), (b), and (c) follow immediately from Theorem 3.5.

(d) Let $h = \sum_{k=1}^n h_{k-1}\chi_{[t_{k-1}, t_k)}$ with $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t$, $h_{k-1} \in L^2(\mathscr{C}_{t_{k-1}})$ be a simple process in $\mathfrak{H}[t_0, t]$. Then, for any $g \in L^2(\mathscr{C}_{t_0})$, we have

$$\begin{aligned} m\left(g \int_{t_0}^t h\, d\Psi_s\right) &= \sum_{k=1}^n m(g h_{k-1} \Delta\Psi_k) \\ &= 0 \qquad \text{by Lemma 3.4.} \end{aligned}$$

It follows that $M_{t_0}(\int_{t_0}^t h\, d\Psi_s) = 0$ for all simple processes h in $\mathfrak{H}[t_0, t]$, and hence for all $h \in \mathfrak{H}[t_0, t]$.

(e) Let $0 \leqslant t_0 \leqslant t$. Then

$$\begin{aligned} M_{t_0}\left(\int_0^t f\, d\Psi\right) &= M_{t_0}\left(\int_0^{t_0} f\, d\Psi + \int_{t_0}^t f\, d\Psi\right) \\ &= M_{t_0}\left(\int_0^{t_0} f\, d\Psi\right) \quad \text{by (d)} \\ &= \int_0^{t_0} f\, d\Psi. \end{aligned}$$

Q.E.D.

Remark. For continuous f, one might be tempted to try to define the stochastic integral $\int_{t_0}^t f\, d\Psi_s$ as the limit of $\sum_{k=1}^n f(\tau_k)(\Psi_{t_k} - \Psi_{t_{k-1}})$ for $t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \cdots \leqslant \tau_n \leqslant t_n = t$ as the partition of $[t_0, t]$ becomes finer. However, just as for the stochastic integral with respect to a Wiener process, the limit will depend on the choice of $\tau_k \in [t_{k-1}, t_k]$; $\tau_k = t_{k-1}$ giving the Itô-integral. To see this, consider $f(s) = \Psi_s = \Psi(u\chi_{[0,s]})$ and suppose that $u(x) = 1$ for all $x \in \mathbb{R}_+$, and set $\tau_k = (1-\lambda)\, t_{k-1} + \lambda t_k$, where $0 \leqslant \lambda \leqslant 1$. (Note that $\Psi_s \in \mathfrak{H}[t_0, t]$.) Put

$$S_n = \sum_{k=1}^n \Psi_{\tau_k}(\Psi_{t_k} - \Psi_{t_{k-1}}).$$

Then

$$S_n = \sum_{k=1}^{n} \Psi_{t_{k-1}}(\Psi_{t_k} - \Psi_{t_{k-1}}) + \sum_{k=1}^{n} \Psi(\chi_{[t_{k-1},\tau_k]})^2$$
$$+ \sum_{k=1}^{n} \Psi(\chi_{[t_{k-1},\tau_k]})\, \Psi(\chi_{[\tau_k,t_k]}).$$

Using independence and the anti-commutation relations, we see that the last term has L^2-norm equal to $\{\sum_{k=1}^{n} (\tau_k - t_{k-1})(t_k - \tau_k)\}^{1/2} \leqslant \{\sum_{k=1}^{n} (\tau_k - t_{k-1})(t_k - t_{k-1})\}^{1/2}$ which converges to zero as the partition becomes finer. The second term is equal to $\sum_{k=1}^{n} (\tau_k - t_{k-1}) \mathbb{1} = \lambda(t - t_0) \mathbb{1}$. Hence we obtain

$$S_n \to \int_{t_0}^{t} \Psi\, d\Psi_s + \lambda(t - t_0) \mathbb{1}.$$

We see that the limit depends on λ. Evidently, it is only for $\lambda = 0$ that the limit is a martingale. (The Stratonovich–Clifford stochastic integral would be obtained as the limit of

$$\sum \tfrac{1}{2}(\Psi_{t_{k-1}} + \Psi_{t_k})(\Psi_{t_k} - \Psi_{t_{k-1}}) = \int_{t_0}^{t} \Psi\, d\Psi_s + \tfrac{1}{2}(t - t_0) \mathbb{1}.)$$

Let $\mathfrak{H}_{\mathrm{loc}}[0, \infty)$ denote the set of processes f such that $f \in \mathfrak{H}[0, t]$ for all $t \in \mathbb{R}_+$. Then we have seen that $\{\int_0^t f\, d\Psi_s : t \in \mathbb{R}_+\}$ is a centred L^2-martingale. We shall show, in the next section, that every centred L^2-martingale has this form. We end this section by showing that the submartingale $(\int_0^t f\, d\Psi_s)^* (\int_0^t f\, d\Psi_s)$ can be decomposed as the sum of a martingale and an increasing positive process.

Let $f \in L^1([t_0, t], d\mu; L^1(\mathscr{C}))$, the Banach space of measurable maps f from $[t_0, t]$ into $L^1(\mathscr{C})$ such that $\|f(\cdot)\|_1$ is integrable over $[t_0, t]$ with respect to $d\mu$. Then $\int_{t_0}^t f(s)\, d\mu(s)$ is a well-defined element of $L^1(\mathscr{C})$. Indeed, it is uniquely determined by the formula

$$m\left(g \int_{t_0}^{t} f(s)\, d\mu(s)\right) = \int_{t_0}^{t} m(gf(s))\, d\mu(s)$$

for all $g \in L^\infty(\mathscr{C})$.

Furthermore, for any $f \in L^1([t_0, t], d\mu; L^1(\mathscr{C}))$ and $\alpha \in \mathbb{R}_+$, we have

$$\int_{t_0}^{t} M_\alpha(f(s))\, d\mu(s) = M_\alpha \left(\int_{t_0}^{t} f(s)\, d\mu(s)\right).$$

In particular, it follows that $\int_{t_0}^t M_{t_0}(f(s))\, d\mu(s) \in L^1(\mathscr{C}_{t_0})$.

3.13 DEFINITION. An element $h \in L^2(\mathscr{C})$ is said to be even (resp. odd) if $\beta h = h$ (resp. $-h$), where $\beta: L^2(\mathscr{C}) \to L^2(\mathscr{C})$ denotes the self-adjoint unitary $D^{-1}\Gamma(-1)\,D$. We say that h has definite parity if h is either even or odd. If $f: [t_0, t] \to L^2(\mathscr{C})$, we say that f is even (resp. odd) if $\beta f(s) = f(s)$ (resp. $-f(s)$) for all $s \in [t_0, t]$. We say that f has definite parity if f is either even or odd.

Writing $h \in L^2(\mathscr{C})$ as $h = \frac{1}{2}(h + \beta h) + \frac{1}{2}(h - \beta h)$, we see that h is (uniquely) the sum of an even element and an odd element of $L^2(\mathscr{C})$ (defining 0 as both even and odd).

Furthermore, $\beta h = \pm h$ iff $\Gamma(-1)\,h\Omega = \pm h\Omega$, and so h is even iff $h\Omega \in \bigoplus_{n=0}^{\infty} \Lambda_{2n}(\mathscr{H})$, and h is odd iff $h\Omega \in \bigoplus_{n=0}^{\infty} \Lambda_{2n+1}(\mathscr{H})$. It follows that if h is even (resp. odd) there is a sequence (g_n) of even (resp. odd) polynomials in $\mathscr{C}^0$, the algebraic span of the fields $\Psi(v)$, $v \in \mathscr{H}$, such that $g_n\Omega \to h\Omega$ in $\Lambda(\mathscr{H})$; i.e., $g_n \to h$ in $L^2(\mathscr{C})$. Evidently, for any $v \in \mathscr{H}$, $\Psi(v)$ is odd. We also note that if $h \in L^2(\mathscr{C}_t)$, then $\beta h \in L^2(\mathscr{C}_t)$, and if $g \in L^\infty(\mathscr{C}_t)$, then $\beta g \in L^\infty(\mathscr{C}_t)$.

3.14 LEMMA. (i) *Let $h' \in L^2(\mathscr{C})$ be even, and $h'' \in L^2(\mathscr{C})$ be odd. Then $m(h'^* h'') = 0$.*

(ii) *Let $h \in L^2(\mathscr{C})$, and $g \in L^\infty(\mathscr{C})$ have definite parity. Then hg is even if h and g have the same parity, otherwise hg is odd.*

Proof. (i) $m(h'^* h'') = (h'\Omega, h''\Omega) = 0$, since $h'\Omega$ and $h''\Omega$ lie in orthogonal subspaces of $\Lambda(\mathscr{H})$.

(ii) Suppose $h \in L^2(\mathscr{C})$, $g \in L^\infty(\mathscr{C})$ are both even. Then there is a sequence (g_n) of even polynomials in $\mathscr{C}^0$ such that $g_n \to h$ in $L^2(\mathscr{C})$, and therefore $g_n g \to hg$ in $L^2(\mathscr{C})$. But $\Gamma(-1)\, g_n g\Omega = g_n g\Omega$ for all n, and so $\Gamma(-1)\, hg\Omega = hg\Omega$; i.e., $\beta(hg) = hg$. The other cases are similar. Q.E.D.

3.15 LEMMA. *Let $0 \leqslant t_0 \leqslant s \leqslant t$, and suppose $g \in L^2(\mathscr{C}_{t_0})$ has definite parity. Then*

$$(\Psi_t - \Psi_s)\, g = \pm g(\Psi_t - \Psi_s)$$

depending on whether g is even or odd.

Proof. Suppose g is odd. Then there is a sequence (g_n) of odd polynomials in $\mathscr{C}^0_{t_0}$, the self-adjoint subalgebra of $\mathscr{C}^0$ generated by the fields $\Psi(v)$ with $v \in \mathscr{H}$ with ess supp $v \subseteq [0, t_0]$, such that $g_n \to g$ in $L^2(\mathscr{C}_{t_0})$.

Now, $\Psi_t - \Psi_s = \Psi(u\chi_{[s,t]})$ and $u\chi_{[s,t]}$ is orthogonal in $\mathscr{H} = L^2(\mathbb{R}_+, ds)$ to all v with ess supp $v \subseteq [0, t_0]$. It follows from the anti-commutation relations that

$$(\Psi_t - \Psi_s)\, g_n = -g_n(\Psi_t - \Psi_s)$$

The result follows by letting $n \to \infty$. For g even the proof is similar. Q.E.D.

3.16 THEOREM. *Let $f', f'' \in \mathfrak{H}[t_0, t]$ both have definite parity. Then*

$$M_{t_0}\left(\left(\int_{t_0}^{t} f'\, d\Psi\right)^* \left(\int_{t_0}^{t} f''\, d\Psi\right)\right) = \pm \int_{t_0}^{t} M_{t_0}(f'(s)^* f''(s))\, d\mu(s)$$

depending on whether f' and f'' have equal or opposite parity.

Proof. First we note that if $f \in \mathfrak{H}[t_0, t]$, then $|f(\cdot)|^2 \in L^1([t_0, t], d\mu; L^1(\mathscr{C}))$ and so, by polarization, we see that $f'(\cdot)^* f''(\cdot)$ is (a process) in $L^1([t_0, t], d\mu; L^1(\mathscr{C}))$. Hence $\int_{t_0}^{t} M_{t_0}(f'(s)^* f''(s))\, d\mu(s) \in L^1(\mathscr{C}_{t_0})$.

We shall prove the theorem for simple processes, and then use a limiting argument. Let $h', h'' \in \mathfrak{H}[t_0, t]$ be simple processes of definite parity. Then there is a partition $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t$ such that

$$h' = \sum_{k=1}^{n} h'_{k-1} \chi_{[t_{k-1}, t_k)} \qquad \text{and} \qquad h'' = \sum_{k=1}^{n} h''_{k-1} \chi_{[t_{k-1}, t_k)}$$

on $[t_0, t)$ for $h'_{k-1}, h''_{k-1} \in L^2(\mathscr{C}_{t_{k-1}})$.

For any $g \in L^\infty(\mathscr{C}_{t_0})$, we have

$$\begin{aligned} m(M_{t_0}(I(h')^* I(h''))\, g) &= m(I(h')^* I(h'')\, g) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} m(\Delta\Psi_k h'^*_{k-1} h''_{j-1} \Delta\Psi_j\, g). \end{aligned}$$

By Lemma 3.4, we see that the off-diagonal terms all vanish, and if h' and h'' have opposite parity and g is even then, by Lemma 3.14, the diagonal terms also vanish. For g odd, we apply Lemma 3.15 to obtain

$$-\sum_{k=1}^{n} m(h'^*_{k-1} h''_{k-1}\, g(\Delta\Psi_k)^2) = -\sum_{k=1}^{n} m(h'^*_{k-1} h''_{k-1}\, g) \int_{t_{k-1}}^{t_k} d\mu.$$

Thus, for h', h'' of opposite parity and g odd, we have

$$m(I(h')^* I(h'')\, g) = -\int_{t_0}^{t} m(h'(s)^*\, h''(s)\, g)\, d\mu(s).$$

However, this remains valid for g even since then both sides vanish. Therefore it holds for all $g \in L^\infty(\mathscr{C}_{t_0})$. Hence, for h', h'' of opposite parity,

$$M_{t_0}(I(h')^* I(h'')) = -\int_{t_0}^{t} M_{t_0}(h'(s)^*\, h''(s))\, d\mu(s).$$

For h', h'' of equal parity the proof is similar.

Now let f', $f'' \in \mathfrak{H}[t_0, t]$ be of definite parity, and let (h'_n), (h''_n) be sequences of simple processes converging to f', f'', respectively, in $\mathfrak{H}[t_0, t]$.

By considering $\frac{1}{2}(h_n' \pm \beta h_n')$ and $\frac{1}{2}(h_n'' \pm \beta h_n'')$ as necessary, we may suppose that h_n' and h_n'' have the same parity as f' and f'', respectively.

We have $I(h_n) \to I(f')$ and $I(h_n'') \to I(f'')$ in $L^2(\mathscr{C})$ and so $I(h_n')^* I(h_n'') \to I(f')^* I(f'')$ in $L^1(\mathscr{C})$. Hence $M_{t_0}(I(h_n')^* I(h_n'')) \to M_{t_0}(I(f')^* I(f''))$ in $L^1(\mathscr{C})$.

Now let $g \in L^\infty(\mathscr{C})$. Then

$$\begin{aligned}
&\left| \int_{t_0}^t m((f'(s)^* f''(s) - h_n'(s)^* h_n''(s))\, g)\, d\mu(s) \right| \\
&\quad = \left| \int_{t_0}^t m(f'(s)^* (f''(s) - h_n''(s))\, g + (f'(s)^* - h_n'(s)^*)\, h_n''(s)\, g)\, d\mu \right| \\
&\quad \leqslant \int_{t_0}^t \| g f'(s)^* \|_2 \, \| f''(s) - h_n''(s) \|_2 \, d\mu \\
&\qquad + \int_{t_0}^t \| f'(s)^* - h_n'(s)^* \|_2 \, \| h_n''(s)\, g \|_2 \, d\mu \\
&\quad \leqslant \left\{ \int_{t_0}^t \| f'(s)^* \|_2^2 \, d\mu \int_{t_0}^t \| f''(s) - h_n''(s) \|_2^2 \, d\mu \right\}^{1/2} \| g \|_\infty \\
&\qquad + \left\{ \int_{t_0}^t \| f'(s) - h_n'(s) \|_2^2 \, d\mu \int_{t_0}^t \| h_n''(s) \|_2^2 \, d\mu \right\}^{1/2} \| g \|_\infty .
\end{aligned}$$

Taking the supremum over $g \in L^\infty(\mathscr{C})$ with $\| g \|_\infty \leqslant 1$ and letting $n \to \infty$, we conclude that

$$\int_{t_0}^t h_n'(s)^* h_n''(s)\, d\mu \to \int_{t_0}^t f'(s)^* f''(s)\, d\mu$$

in $L^1(\mathscr{C})$, and so $M_{t_0}(\int_{t_0}^t h_n'(s)^* h_n''(s)\, d\mu) \to M_{t_0}(\int_{t_0}^t f'(s)^* f''(s)\, d\mu)$ in $L^1(\mathscr{C})$.

Hence

$$\begin{aligned}
M_{t_0}(I(f')^* I(f'')) &= L^1 - \lim M_{t_0}(I(h_n')^* I(h_n'')) \\
&= \pm L^1 - \lim \int_{t_0}^t M_{t_0}(h_n'(s)^* h_n''(s))\, d\mu \\
&= \pm \int_{t_0}^t M_{t_0}(f'(s)^* f''(s))\, d\mu. \qquad \text{Q.E.D.}
\end{aligned}$$

3.17 THEOREM. *Let $f \in \mathfrak{H}[t_0, t]$. Then*

$$M_{t_0}\left(\left| \int_{t_0}^t f\, d\Psi \right|^2 \right) = \int_{t_0}^t M_{t_0}(|\beta f(s)|^2)\, d\mu.$$

Proof. It is easy to see that $\beta f \in \mathfrak{H}[t_0, t]$ and so we can write $f = f_1 + f_2$ with $f_1, f_2 \in \mathfrak{H}[t_0, t]$, f_1 even and f_2 odd. (In fact $f_1 = \frac{1}{2}(f + \beta f)$ and $f_2 = \frac{1}{2}(f - \beta f)$.)

By Theorem 3.16, we have

$$\begin{aligned} M_{t_0}(I(f_i)^* I(f_j)) &= (2\delta_{ij} - 1) \int_{t_0}^{t} M_{t_0}(f_i(s)^* f_j(s))\, d\mu \\ &= \int_{t_0}^{t} M_{t_0}((\beta f_i(s))^* \beta f_j(s))\, d\mu. \end{aligned}$$

Summing over $i, j = 1, 2$ gives the desired result. Q.E.D.

3.18 THEOREM. *Let $f \in \mathfrak{H}_{\text{loc}}[0, \infty)$. Then*

$$Z_t = \left| \int_0^t f\, d\Psi_s \right|^2 - \int_0^t |\beta f(s)|^2\, d\mu, \qquad t \in \mathbb{R}_+,$$

defines a centred L^1-martingale adapted to the family $\{\mathscr{C}_t : t \in \mathbb{R}_+\}$. In particular, $\{|\int_0^t f\, d\Psi|^2 : t \in \mathbb{R}_+\}$ can be written as the sum of a centred L^1-martingale and a positive increasing process.

Proof. We have already established that $Z_t \in L^1(\mathscr{C}_t)$ for all $t \in \mathbb{R}_+$. For $0 \leqslant t_0 \leqslant t$, write

$$\left| \int_0^t f\, d\Psi \right|^2 = \left(\int_0^{t_0} f\, d\Psi + \int_{t_0}^{t} f\, d\Psi \right)^* \left(\int_0^{t_0} f\, d\Psi + \int_{t_0}^{t} f\, d\Psi \right).$$

By Theorem 3.12, we have

$$M_{t_0}\left(\left(\int_0^{t_0} f\, d\Psi \right)^* \left(\int_{t_0}^{t} f\, d\Psi \right) \right) = \left(\int_0^{t_0} f\, d\Psi \right)^* M_{t_0}\left(\int_{t_0}^{t} f\, d\Psi \right) = 0,$$

and similarly

$$M_{t_0}\left(\left(\int_{t_0}^{t} f\, d\Psi \right)^* \left(\int_0^{t_0} f\, d\Psi \right) \right) = M_{t_0}\left(\int_{t_0}^{t} f\, d\Psi \right)^* \int_0^{t_0} f\, d\Psi = 0.$$

Therefore

$$\begin{aligned} M_{t_0}\left(\left| \int_0^t f\, d\Psi \right|^2 \right) &= \left| \int_0^{t_0} f\, d\Psi \right|^2 + M_{t_0}\left(\left| \int_{t_0}^{t} f\, d\Psi \right|^2 \right) \\ &= \left| \int_0^{t_0} f\, d\Psi \right|^2 + M_{t_0}\left(\int_{t_0}^{t} |\beta f(s)|^2\, d\mu \right) \end{aligned}$$

by Theorem 3.17,

$$\begin{aligned}
&= \left| \int_0^{t_0} f\, d\Psi \right|^2 + M_{t_0}\left(\int_0^t |\beta f(s)|^2\, d\mu \right) \\
&\quad - M_{t_0}\left(\int_0^{t_0} |\beta f(s)|^2\, d\mu \right) \\
&= \left| \int_0^{t_0} f\, d\Psi \right|^2 + M_{t_0}\left(\int_0^t |\beta f(s)|^2\, d\mu \right) \\
&\quad - \int_0^{t_0} |\beta f(s)|^2\, d\mu,
\end{aligned}$$

i.e., $M_{t_0}(Z_t) = Z_{t_0}$.

In particular, $m(Z_t) = m(M_0(Z_t)) = m(Z_0) = 0$. Hence $\{Z_t \colon t \in \mathbb{R}_+\}$ is a centred L^1-martingale. Q.E.D.

Thus we see that the submartingale $\{|\int_0^t f\, d\Psi_s|^2\}$ has a decomposition of Doob–Meyer type.

We define the pointed bracket process, $\langle \int_0^t f\, d\Psi, \int_0^t g\, d\Psi \rangle$ by polarization from the quadratic form

$$\left\langle \int_0^t f\, d\Psi, \int_0^t f\, d\Psi \right\rangle = \int_0^t |\beta f(s)|^2\, d\mu,$$

i.e.,

$$\left\langle \int_0^t f\, d\Psi, \int_0^t g\, d\Psi \right\rangle = \int_0^t (\beta f(s))^* \, (\beta g(s))\, d\mu.$$

We shall see that any L^2-martingale is a stochastic integral (Section 4). Hence we can define the bracket between any two L^2-martingales.

4. The Representation of Martingales as Stochastic Integrals

We shall prove here that every centred L^2-martingale adapted to $\{\mathscr{C}_t \colon t \in \mathbb{R}_+\}$ is given by an Itô–Clifford stochastic integral. Throughout this section Ψ_s will denote $\Psi(\chi_{[0,s]})$, $s \in \mathbb{R}_+$ (in other words, we have chosen $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ to be $u(s) = 1 \forall s \in \mathbb{R}_+$). The measure $d\mu(s)$ is now just ds.

If w is a function on $\mathbb{R}^n$, we shall use the notation $\hat{w}$ for the function $\hat{w}(x_1,\dots,x_n)=\theta(x_2-x_1)\cdots\theta(x_n-x_{n-1})\,w(x_1,\dots,x_n)$, where $\theta(x)$ is the Heaviside step-function.

4.1 THEOREM. *Let $\{X_t\colon t\in\mathbb{R}_+\}$ be a centred L^2-martingale adapted to the family $\{\mathscr{C}_t\colon t\in\mathbb{R}_+\}$. Then there is $f\in\mathfrak{H}_{\mathrm{loc}}[0,\infty)$ such that $X_t=\int_0^t f(s)\,d\Psi_s$ for all $t\in\mathbb{R}_+$, where $\Psi_s=\Psi(\chi_{[0,s]})$.*

Proof. For each $t\in\mathbb{R}_+$, X_t can be written as a (direct) sum $X_t=\sum_{n=1}^{\infty}X_t^{(n)}$, where $X_t^{(n)}\in L^2(\mathscr{C}_t)$ and $X_t^{(n)}\Omega\in\Lambda_n(\mathscr{H})$. (There is no $n=0$ term since X_t is centred.) Since $\Gamma(e_s)\colon\Lambda_n(\mathscr{H})\to\Lambda_n(\mathscr{H})$, it is clear that each $\{X_t^{(n)}\colon t\in\mathbb{R}_+\}$ is also a centred L^2-martingale.

Suppose we know that there is $f_n\in\mathfrak{H}_{\mathrm{loc}}[0,\infty)$ such that $X_t^{(n)}=\int_0^t f_n\,d\Psi_s$. By the isometry property, we have $(X_t^{(n)},X_t^{(k)})_{L^2}=\int_0^t (f_n(s),f_k(s))_{L^2}\,ds$. In particular, f_n and f_k are orthogonal in each $\mathfrak{H}[0,t]$ for $n\neq k$. Furthermore,

$$\|X_t\|_2^2=\sum_{n=1}^{\infty}\|X_t^{(n)}\|_2^2=\sum_{n=1}^{\infty}\|f_n\|_{\mathfrak{H}[0,t]}^2$$

and so we see that $f(s)=\sum_{n=1}^{\infty}f_n(s)$ is a well-defined element of $\mathfrak{H}[0,t]$ for each $t\in\mathbb{R}_+$, and $X_t=\sum_{n=1}^{\infty}X_t^{(n)}=\sum_{n=1}^{\infty}\int_0^t f_n\,d\Psi_s=\int_0^t f\,d\Psi_s$ by the isometry property.

It is enough, therefore, to prove the theorem for an "n-particle martingale": i.e., we may suppose that $X_t\Omega\in\Lambda_n(\mathscr{H})$ for all $t\in\mathbb{R}_+$, for some $n\geqslant 1$. Then there is an anti-symmetric function w in $L^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that $X_t=\int w(x_1,\dots,x_n)\,\chi_{[0,t]^n}(x_1,\dots,x_n)\colon\psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n$. Fix $t>0$. Suppose first that $w\in\mathscr{S}(\mathbb{R}^n)$, with $\operatorname{supp}w\subseteq[0,t]^n$, and w vanishes in a neighbourhood of the set $\{(x_1,\dots,x_n)\in\mathbb{R}^n\colon x_i=x_j \text{ some } i\neq j\}$.

Then $\hat{w}$ also belongs to $\mathscr{S}(\mathbb{R}^n)$. For $0\leqslant s\leqslant t$, set $f(s)=n!\int\hat{w}(x_1,\dots,x_{n-1},s)\colon\psi(x_1)\cdots\psi(x_{n-1})\colon dx_1\cdots dx_{n-1}$. Then $\hat{w}(\cdot,\dots,s)$ has support in $[0,s]^{n-1}$ and so $f(s)\in L^2(\mathscr{C}_s)$. (Indeed, $f(s)$ is in the closure of the linear span of Wick monomials of degree $n-1$ in $L^2(\mathscr{C}_s)$.) Furthermore,

$$\begin{aligned}\|f(s)-f(s')\|_2^2&=\|f(s)\,\Omega-f(s')\,\Omega\|^2\\&=\frac{(n!)^2}{(n-1)!}\int\left|\left(\prod_{i=1}^{n-1}\theta(s-x_i)\right)w(x_1,\dots,s)\right.\\&\qquad\left.-\left(\prod_{j=1}^{n-1}\theta(s'-x_j)\right)w(x_1,\dots,s')\right|^2dx_1\cdots dx_{n-1}.\end{aligned}$$

It follows that the map $s\mapsto f(s)\colon[0,t]\to L^2(\mathscr{C})$ is continuous, and hence $f\in\mathfrak{H}[0,t]$.

The stochastic integral of f is given as

$$\int_0^t f(s)\,d\Psi_s = L^2 - \lim_{k\to\infty} \sum_{j=1}^k f(t_{j-1})(\Psi_{t_j} - \Psi_{t_{j-1}}),$$

where $t_j = jt/k$.

Now $f(t_{j-1})(\Psi_{t_j} - \Psi_{t_{j-1}})\,\Omega = f(t_{j-1})\,\Psi(\chi_{[t_{j-1},t_j]})\,\Omega$ which belongs to $\Lambda_n(\mathscr{H})$, since $\int w(x_1,\dots,x_{n-1},t_{j-1})\,\chi_{[t_{j-1},t_j]}(x_i)\,dx_i = 0$ for any $1 \leqslant i \leqslant n-1$.

Let $v \in \Lambda(\mathscr{H})$ and consider $(v, \int_0^t f(s)\,d\Psi_s\Omega)$. By the previous remark, we may assume that $v \in \Lambda_n(\mathscr{H})$.

We have (where $\mathfrak{P}_n$ denotes the permutation group on n letters)

$$\begin{aligned}
&\left(v, \int_0^t f(s)\,d\Psi_s\Omega\right)\\
&= \lim_{k\to\infty} \sum_{j=1}^k (v, f(t_j)(\Psi_{t_j} - \Psi_{t_{j-1}})\,\Omega)\\
&= \lim_{k\to\infty} \sum_{j=1}^k \sqrt{n!} \int \bar{v}(x_1,\dots,x_n) \sum_{\pi\in\mathfrak{P}_n} (-1)^\pi\,\hat{w}(x_{\pi(1)},\dots,x_{\pi(n-1)},t_j)\\
&\quad \times \chi_{[t_{j-1},t_j]}(x_{\pi(n)})\,dx_1\cdots dx_n\\
&= \sqrt{n!} \int \bar{v}(x_1,\dots) \sum_\pi \theta(x_{\pi(2)} - x_{\pi(1)}) \cdots \theta(x_{\pi(n)} - x_{\pi(n-1)})(-1)^\pi\\
&\quad \times w(x_{\pi(1)},\dots,x_{\pi(n)})\,dx_1\cdots dx_n\\
&= \sqrt{n!} \int \bar{v}(x_1,\dots,x_n)\,w(x_1,\dots,x_n)\,dx_1\cdots dx_n\\
&= \left(v, \int w(x_1,\dots,x_n)\colon \psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n\Omega\right).
\end{aligned}$$

It follows that $\int w(x_1,\dots,x_n)\colon \psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n\Omega = \int_0^t f(s)\,d\Psi_s\Omega$, and so

$$\int w(x_1,\dots,x_n)\colon \psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n = \int_0^t f(s)\,d\Psi_s.$$

Now let $w \in L^2_{\mathrm{loc}}(\mathbb{R}^n_+)$ be anti-symmetric. Then there is a sequence (w_k) in $\mathscr{S}(\mathbb{R}^n)$ such that each w_k is anti-symmetric, vanishes in a neighbourhood of $\{(x_1,\dots,x_n) \in \mathbb{R}^n\colon x_i = x_j \text{ some } i \neq j\}$ (depending on k), and has support in $[0,t]^n$, and such that $w_k \to w\chi_{[0,t]^n}$ in $L^2(\mathbb{R}^n_+)$. It follows that $\int w_k\colon \psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n\Omega$ converges to $\int w(x_1,\dots)\,\chi_{[0,t]^n}(x_1,\dots)\colon \psi(x_1)\cdots\colon dx_1\cdots dx_n\Omega$ in $\Lambda(\mathscr{H})$, and so $\int w_k\colon \psi(x_1)\cdots\colon dx_1\cdots dx_n$ converges to $\int w(x_1,...)\,\chi_{[0,t]^n}(x_1,...)\colon \psi(x_1)\cdots\colon dx_1\cdots dx_n$ in $L^2(\mathscr{C})$. Putting $f_k(s) = n!\int \hat{w}_k(x_1,..., x_{n-1}, s)\colon \psi(x_1)\cdots\psi(x_{n-1})\colon dx_1\cdots dx_{n-1}$, we have $f_k \in \mathfrak{H}[0,t]$ and

$$\int w_k(x_1,..., x_n)\colon \psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n = \int_0^t f_k(s)\, d\Psi_s.$$

Now, $w_k \to w\chi_{[0,t]^n}$ in $L^2(\mathbb{R}_+^n)$ implies that

$$\int |\theta(s-x_1)\cdots\theta(s-x_{n-1})(w_{n_k} - w\chi_{[0,t]^n})(x_1,..., s)|^2\, dx_1\cdots dx_{n-1} \to 0$$

for almost all s, for some subsequence (w_{n_k}).

Hence

$$\begin{aligned}
&\|f_{n_k}(s)\,\Omega - f_{n_j}(s)\,\Omega\|^2\\
&= \frac{(n!)^2}{(n-1)!}\int |\theta(s-x_1)\cdots\theta(s-x_{n-1})(w_{n_k} - w_{n_j})(x_1,..., s)|^2\, dx_1\cdots dx_{n-1}\\
&\to 0 \qquad \text{for almost all } s;
\end{aligned}$$

i.e., $(f_{n_k}(s))$ is $L^2(\mathscr{C})$-Cauchy for almost all s.

It follows that there is $f(s) \in L^2(\mathscr{C})$ such that $f_{n_k}(s) \to f(s)$ in $L^2(\mathscr{C})$ for almost all s. Moreover, since $f_{n_k}(s) \in L^2(\mathscr{C}_s)$, it follows that $f(s) \in L^2(\mathscr{C}_s)$ for almost all s. Thus $f(\cdot)$ is an L^2-process on $[0,t]$. For any $g \in L^2(\mathscr{C})$, $(g, f_{n_k}(s))$ is continuous in s and so $(g, f(s)) = \lim (g, f_{n_k}(s))$ a.e. is (Borel) measurable in s. Hence $f(\cdot)$ is measurable. Furthermore,

$$\begin{aligned}
\|f(s)\|_2^2 &= \|f(s)\,\Omega\|^2\\
&= \frac{(n!)^2}{(n-1)!}\|\theta(s-\cdot)\cdots\theta(s-\cdot)w(\cdot,..., s)\,\chi_{[0,t]}(\cdot,..., s)\|^2_{L^2(\mathbb{R}_+^{n-1})}
\end{aligned}$$

which is integrable over $[0,t]$ with respect to ds. Thus $f \in \mathfrak{H}[0,t]$. Moreover, we have

$$\begin{aligned}
\|f - f_{n_k}\|^2_{\mathfrak{H}[0,t]} &\leqslant \frac{(n!)^2}{(n-1)!}\|w\chi_{[0,t]} - w_{n_k}\|^2_{L^2(\mathbb{R}_+^n)}\\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|\int_0^t f\, d\Psi_s - \int_0^t f_{n_k}\, d\Psi_s\right\|_2^2 &= \|f - f_{n_k}\|^2_{\mathfrak{H}[0,t]}\\
&\to 0.
\end{aligned}$$

Therefore

$$\begin{aligned}\int w(x_1,\ldots,x_n)\,\chi_{[0,t]^n}(x_1,\ldots,x_n)\colon \psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n \\ &= \lim \int w_{n_k}(x_1,\ldots,x_n)\colon \psi(x_1)\cdots\psi(x_n)\colon dx_1\cdots dx_n \\ &= \lim \int_0^t f_{n_k}\,d\Psi_s \\ &= \int_0^t f\,d\Psi_s;\end{aligned}$$

i.e., for fixed $t > 0$, $X_t = \int_0^t f\,d\Psi_s$ with $f \in \mathfrak{H}[0, t]$. Similar, for $t' > 0$ we obtain

$$X_{t'} = \int_0^{t'} f'\,d\Psi_s \qquad \text{for some } f' \in \mathfrak{H}[0, t'].$$

But if $t' > t$, we have

$$\begin{aligned}\int_0^t f\,d\Psi_s = X_t = M_t(X_{t'}) = M_t\left(\int_0^{t'} f'\,d\Psi_s\right) \\ = \int_0^t f'\,d\Psi_s.\end{aligned}$$

It follows that $f'(s) = f(s)$ a.e. on $[0, t]$, and so there is $f \in \mathfrak{H}_{\rm loc}[0, \infty)$ such that $X_t = \int_0^t f(s)\,d\Psi_s$ for all $t \in \mathbb{R}_+$. Q.E.D.

As a corollary we can write down an expression for $\int_0^t \Psi\,d\Psi_s$ in terms of Wick products. Indeed, let $w(x_1, x_2) = \frac{1}{2}(\theta(x_2 - x_1) - \theta(x_1 - x_2))\,\theta(x_1)\,\theta(x_2)$. Then $w \in L^2_{\rm loc}(\mathbb{R}^2_+)$ and is anti-symmetric. Furthermore, $\hat{w}(x_1, x_2) = \frac{1}{2}\theta(x_2 - x_1)\,w(x_1, x_2) = \frac{1}{2}\theta(x_2 - x_1)\,\theta(x_1)\,\theta(x_2)$. Setting $f(s) = 2! \int \hat{w}(x, s)\,\psi(x)\,dx$, we have

$$\begin{aligned}f(s) = \int \theta(s - x)\,\theta(x)\,\psi(x)\,dx = \int \chi_{[0,s]}(x)\,\psi(x)\,dx \\ = \Psi(s).\end{aligned}$$

Thus

$$\begin{aligned}\int_0^t f(s)\,d\Psi_s = \int_0^t \Psi(s)\,d\Psi_s \\ = \int w(x_1, x_2)\,\chi_{[0,t]^2}\colon \psi(x_1)\,\psi(x_2)\colon dx_1\,dx_2.\end{aligned}$$

5. The Stochastic Integral with Respect to a Wick Martingale

We have constructed the stochastic integral with respect to the field $\Psi_s = \Psi(u\chi_{[0,s]})$, $u \in L^2_{\text{loc}}(\mathbb{R}_+)$ and we shall show now that precisely the same method can be employed to define a stochastic integral with respect to a Wick monomial martingale $W(u_1,..., u_n; s)$; $u_1,..., u_n \in L^2_{\text{loc}}(\mathbb{R}_+)$.

Let $u_1,..., u_n \in L^2_{\text{loc}}(\mathbb{R}_+)$ be fixed and denote $W(u_1,..., u_n; s)$ by W_s; then $\{W_s: s \in \mathbb{R}_+\}$ is an L^∞-martingale adapted to $\{\mathscr{C}_s: s \in \mathbb{R}_+\}$.

5.1. Proposition. *For any Wick monomial* $:\Psi(f_1) \cdots \Psi(f_n):$, $f_i \in \mathscr{H}$, $1 \leqslant i \leqslant n$, *we have* $:\Psi(f_1) \cdots \Psi(f_n): :\Psi(f_1) \cdots \Psi(f_n):^* = a\mathbb{1}$ *for some* $a \in \mathbb{R}_+$. *In particular,* $W_s W_s^* = a_s \mathbb{1}$, *where* $s \mapsto a_s$ *is a positive increasing continuous function on* $\mathbb{R}_+$.

Proof. Let $V = \{v_1,..., v_n\}$ be a set of n orthonormal vectors in $\mathscr{H}$. We claim that any product $:\Psi(z_1') \cdots \Psi(z_n'): :\Psi(\bar{z}_1'') \cdots \Psi(\bar{z}_n''):$ with $z_i', z_j'' \in V$, $1 \leqslant i,j \leqslant n$, is a multiple of $\mathbb{1}$. To see this, note that by the anti-symmetry of the Wick product it vanishes if $z_i' = z_j'$, some $i \neq j$ or if $z_k'' = z_l''$ some $k \neq l$. It suffices then, again by anti-symmetry, to consider the case $z_i' = z_i'' = v_i$, $1 \leqslant i \leqslant n$. By Wick's theorem, the orthonormality of the v_i's, and anti-symmetry, we see that $:\Psi(v_1) \cdots \Psi(v_n): :\Psi(\bar{v}_1) \cdots \Psi(\bar{v}_n): = \mathbb{1}$, which verifies our claim.

Using the multilinearity of the map $g_1,..., g_n \mapsto :\Psi(g_1) \cdots \Psi(g_n):$ it follows that $:\Psi(f_1) \cdots \Psi(f_n): :\Psi(f_1) \cdots \Psi(f_n):^* = a\mathbb{1}$ for some $a \in \mathbb{C}$. Since the l.h.s. is non-negative we see that $a \in \mathbb{R}_+$.

The value of a can be computed from

$$\begin{aligned} a &= (\Omega, :\Psi(f_1) \cdots \Psi(f_n): :\Psi(f_1) \cdots \Psi(f_n):^* \Omega) \\ &= \| :\Psi(f_1) \cdots \Psi(f_n):^* \Omega \|^2 \\ &= \| C(\bar{f}_n) \cdots C(\bar{f}_1) \Omega \|^2. \end{aligned}$$

In particular, $W_s W_s^* = a_s \mathbb{1}$, where

$$a_s = \int_0^s \cdots \int_0^s |u(s_1,..., s_n)|^2 \, ds_1 \cdots ds_n$$

with $u(s_1,..., s_n) = (n!)^{1/2} \mathscr{A}(u_n \otimes \cdots \otimes u_1)(s_1,..., s_n)$. Clearly $s \mapsto a_s$ is positive, increasing and continuous. Q.E.D.

Let ν denote the (Borel) measure on $\mathbb{R}_+$ given by the function a_s; i.e., $\nu([\alpha, \beta]) = a_\beta - a_\alpha$ for $\alpha \leqslant \beta \in \mathbb{R}_+$, and let $\mathfrak{H}([t_0, t], d\nu)$ denote the Hilbert-space of processes in $L^2([t_0, t], d\nu; L^2(\mathscr{C}))$.

Let $h = \sum_{k=1}^{n} h_{k-1}\chi_{[t_{k-1},t_k)}$, $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = t$, $h_{k-1} \in L^2(\mathscr{C}_{t_{k-1}})$, be a simple L^2-process on $[t_0, t]$.

5.2 Definition. The stochastic integral of h over $[t_0, t]$ with respect to W_s is

$$\int_{t_0}^{t} h(s)\, dW_s = \sum_{k=1}^{n} h_{k-1}(W_{t_k} - W_{t_{k-1}}).$$

As before, we see that $\int_{t_0}^{t} h(s)\, dW_s$ is independent of the decomposition of h as a sum of step-functions, is linear in h, and belongs to $L^2(\mathscr{C}_t)$. Furthermore, exactly as in Theorem 3.5(b), we see that $\int_{t_0}^{t} h\, dW_s$ is centred.

The analogue of Theorem 3.5(c) is

5.3 Theorem. *For any simple L^2-process h,*

$$\left\| \int_{t_0}^{t} h\, dW_s \right\|_2^2 = \int_{t_0}^{t} \|h(s)\|_2^2\, d\nu(s),$$

that is, the map $h \mapsto \int_{t_0}^{t} h\, dW_s$ is an isometry from $\mathfrak{H}([t_0, t],\ d\nu)$ into $L^2(\mathscr{C}_t)$.

Proof. $\|\int_{t_0}^{t} h\, dW_s\|_2^2 = \sum_{k=1}^{n} \sum_{j=1}^{n} m(\Delta W_j^* h_{j-1}^* h_{k-1} \Delta W_k)$,

(where $\Delta W_k = W_{t_k} - W_{t_{k-1}}$,)

$$= \sum_{k=1}^{n} m(|h_{k-1}|^2\, (W_{t_k} - W_{t_{k-1}})(W_{t_k} - W_{t_{k-1}})^*)$$

by Lemma 3.4,

$$= \sum_{k=1}^{n} m(|h_{k-1}|^2\, (W_{t_k} W_{t_k}^* - W_{t_{k-1}} W_{t_{k-1}}^*))$$

since W_s is a martingale,

$$= \sum_{k=1}^{n} m(|h_{k-1}|^2)(a_{t_k} - a_{t_{k-1}})$$

$$= \int_{t_0}^{t} \|h(s)\|_2^2\, d\nu(s). \qquad \text{Q.E.D.}$$

The construction of $\int_{t_0}^{t} f\, dW_s$ for $f \in \mathfrak{H}([t_0, t], d\nu)$ is carried out exactly as before using simple approximations, and the generalization of Theorem 3.12 holds with ν replacing μ throught. (Note that if the Wick monomials W_s have degree one, then $W_s = \Psi(u\chi_{[0,s]})$ for some $u \in L^2_{\text{loc}}(\mathbb{R}_+)$ and we recover the earlier results.)

The extension of Theorem 3.18 will be considered elsewhere. We note, however, that a result similar to Theorem 3.18 can be easily obtained for the "left-stochastic integral" $\int_0^t dW_s f(s) = (\int_0^t f(s)^* \, dW_s^*)^*$.

Indeed, for a simple L^2-process $h(s)$, we have, for any $g \in L^\infty(\mathscr{C}_{t_0})$,

$$m\left(M_{t_0}\left(\left(\int_{t_0}^t dW_s h(s)\right)^* \left(\int_{t_0}^t dW_s h(s)\right)\right) g\right)$$
$$= \sum_{k=1}^n m(^*_{k-1} \Delta W_k^* \Delta W_k h_{k-1}\, g),$$

since the off-diagonal terms give zero contribution,

$$= \sum_{k=1}^n m(h_{k-1}^*(W_{t_k}^* W_{t_k} - W_{t_{k-1}}^* W_{t_{k-1}})\, h_{k-1}\, g)$$
$$= \int_{t_0}^t m(M_{t_0}(h^*(s)\, h(s))\, g)\, d\nu(s).$$

Hence

$$M_{t_0}\left(\left(\int_{t_0}^t dW_s h(s)\right)^* \left(\int_{t_0}^t dW_s h(s)\right)\right) = \int_{t_0}^t M_{t_0}(h^*(s)\, h(s))\, d\nu(s).$$

By continuity, as in Theorem 3.16, the same result holds with h replaced by any $f \in \mathfrak{H}([t_0, t], d\nu)$. We then obtain

5.4 THEOREM. *For $f \in \mathfrak{H}_{\mathrm{loc}}([0, \infty), d\nu)$,*

$$Z_t = \left|\int_0^t dW_s f\right|^2 - \int_0^t |f(s)|^2\, d\nu$$

defines a centered L^1-martingale adapted to $\{\mathscr{C}_t : t \in \mathbb{R}_+\}$.

6. L^p-PROPERTIES

Let $\Lambda^{(n)}(\mathscr{H}) = \Lambda_0(\mathscr{H}) \oplus \cdots \oplus \Lambda_n(\mathscr{H})$ denote the subspace of $\Lambda(\mathscr{H})$ of vectors with no more than n particles, and let P_n denote the projection of $L^2(\mathscr{C})$ onto $D^{-1}\Lambda^{(n)}(\mathscr{H})$. Then P_n commutes with the conditional expectation M_t on $L^2(\mathscr{C})$.

We recall that the number operator N on $\Lambda(\mathscr{H})$ is defined by $N \restriction \Lambda_n(\mathscr{H}) = n \mathbb{1}_{\Lambda_n(\mathscr{H})}$ for each $n = 0, 1, 2, \ldots$. It is easy to verify that the semigroup $\{e^{-sN}, s \in \mathbb{R}_+\}$ is given by $\{\Gamma(e^{-s}), s \in \mathbb{R}_+\}$. Moreover, it is known that, for any $1 \leqslant p < \infty$, $D^{-1} e^{-sN} D$ defines a bounded map from

$L^2(\mathscr{C})$ into $L^p(\mathscr{C})$ provided s is sufficiently large [11, 31]. By duality, it follows that, for any $1 < p \leqslant \infty$, $D^{-1}e^{-sN}D$ defines a bounded map from $L^p(\mathscr{C})$ into $L^2(\mathscr{C})$ for sufficiently large s.

Now let $f \in L^2(\mathscr{C})$ with $Df \in \Lambda_n(\mathscr{H})$. Then, for any $1 \leqslant p < \infty$, there is $s \in \mathbb{R}_+$ such that $D^{-1}e^{-sN}Df \in L^p(\mathscr{C})$. But $D^{-1}e^{-sN}Df = e^{-sn}f$, and so $f \in L^p(\mathscr{C})$. In other words, $D^{-1}\Lambda_n(\mathscr{H}) \subseteq L^p(\mathscr{C})$ for all $1 \leqslant p < \infty$, and so $P_n L^2(\mathscr{C}) \subseteq L^p(\mathscr{C})$ for all n.

6.1 THEOREM. *Let $f \in \mathfrak{H}([t_0, t], dv)$ and suppose that $f(s) \in P_{N_0} L^2(\mathscr{C})$ v a.e. on $[t_0, t]$, for some integer N_0. Then there is a sequence of simple processes (h_n) with $h_n(s) \in P_{N_0} L^2(\mathscr{C})$ for $t_0 \leqslant s \leqslant t$ such that (h_n) converges to f in $\mathfrak{H}([t_0, t], dv)$.*

Proof. We know that f can be approximated in $\mathfrak{H}([t_0, t], dv)$ by a sequence of simple processes, (g_n), say. Put $h_n(s) = P_{N_0} g_n(s)$ for each $s \in [t_0, t]$. Then h_n is a simple process since $M_s(h_n(s)) = M_s(P_{N_0} g_n(s)) = P_{N_0} M_s(g_n(s)) = P_{N_0} g_n(s) = h_n(s)$. Furthermore,

$$\begin{aligned} \|f(s) - h_n(s)\|_2 &= \|f(s) - P_{N_0} g_n(s)\|_2 \\ &= \|P_{N_0}(f(s) - g_n(s))\|_2 \\ &\leqslant \|f(s) - g_n(s)\|_2 \, v \qquad \text{a.e.} \end{aligned}$$

and so $h_n \to f$ in $\mathfrak{H}([t_0, t], dv)$. Q.E.D.

6.2 THEOREM. *Let $f \in \mathfrak{H}([t_0, t], dv)$ be such that $f(s) \in P_{N_0} L^2(\mathscr{C})$ v a.e. on $[t_0, t]$ for some $N_0 \in \mathbb{N}$. If (h_n) is a sequence of simple processes converging to f in $\mathfrak{H}([t_0, t], dv)$ satisfying $h_n(s) \in P_{N_0} L^2(\mathscr{C})$ v a.e. on $[t_0, t]$, then $\int_{t_0}^t h_n \, dW_s$ converges to $\int_{t_0}^t f \, dW_s$ in $L^p(\mathscr{C})$ as $n \to \infty$, for all $1 \leqslant p < \infty$.*

Proof. By Theorem 6.1, such a sequence (h_n) exists. If $N_1 = N_0 +$ (degree of W_s), we have $\int_{t_0}^t h_n \, dW_s \in P_{N_1} L^2(\mathscr{C})$ and so $\int_{t_0}^t f \, dW_s = L^2(\mathscr{C})-\lim \int_{t_0}^t h_n \, dW_s \in P_{N_1} L^2(\mathscr{C})$. Hence $\int_{t_0}^t h_n \, dW_s \in L^p(\mathscr{C})$ and $\int_{t_0}^t f \, dW_s \in L^p(\mathscr{C})$ for all n, and all $1 \leqslant p < \infty$.

Let $1 \leqslant p < \infty$ be given, and set $X_n = \int_{t_0}^t f \, dW_s - \int_{t_0}^t h_n \, dW_s$. Let $s \in \mathbb{R}_+$ be such that $D^{-1}e^{-sN}D$ is bounded from $L^{p'}(\mathscr{C})$ into $L^2(\mathscr{C})$, where $p' \doteq p/(p-1)$.

For any $g \in L^\infty(\mathscr{C})$ with $\|g\|_{p'} \leqslant 1$, we have

$$\begin{aligned} |m(g^* X_n)| &= |(g, X_n)_{L^2}| \\ &= |(D^{-1}e^{-sN}Dg, D^{-1}e^{sN}DX_n)_{L^2}|, \end{aligned}$$

since DX_n is in the domain of e^{sN},

$$\leqslant \|D^{-1}e^{-sN}Dg\|_2 \|D^{-1}e^{sN}DX_n\|_2$$
$$\leqslant \|D^{-1}e^{-SN}Dg\|_2 e^{sN_1} \|X_n\|_2,$$

since $X_n \in P_{N_1}L^2(\mathscr{C})$,

$$\leqslant Ce^{sN_1} \|g\|_{p'} \|X_n\|_2,$$

for some constant C,

$$\leqslant Ce^{sN_1} \|X_n\|_2.$$

Taking the supremum over all such g, we have

$$\|X_n\|_p \leqslant Ce^{sN_1} \|X_n\|_2 \to 0 \qquad \text{as} \quad n \to \infty. \qquad \text{Q.E.D.}$$

We can strengthen Theorem 2.3 if we know that the martingale X_t belongs to $P_N L^2(\mathscr{C})$ for some N.

6.3 THEOREM. *Let $\{X_t : t \in \mathbb{R}_+\}$ be an L^2-martingale adapted to the family $\{\mathscr{C}_t : t \in \mathbb{R}_+\}$ and suppose that $X_t \in P_N L^2(\mathscr{C})$ for all $t \in \mathbb{R}_+$, for some $N \in \mathbb{N}$. Then, for any $T > 0$, there is a sequence (Y_n) in $\mathscr{W}$ such that $Y_n(t) \to X_t$ in $L^p(\mathscr{C})$, for $1 \leqslant p < \infty$, uniformly in $t \in [0, T]$. Indeed, the sequence (Y_n) may be chosen so that $Y_n(t) \in P_N L^2(\mathscr{C})$ for all $t \in \mathbb{R}_+$ and for all n.*

Proof. Since $X_T \in P_N L^2(\mathscr{C})$, the sequence (w_n) in $\mathfrak{W}$ of Theorem 2.3 can be chosen so that each w_n is a linear combination of Wick monomials of degree at most N; i.e., we can find (w_n) in $P_N\mathfrak{W}$ such that $w_n \to X_T$ in $L^2(\mathscr{C})$.

Putting $Y_n(t) = M_t(w_n)$, we obtain, as in the proof of Theorem 2.3,

$$\|Y_n(t) - X_t\|_2 \leqslant \|w_n - X_T\|_2, \qquad \text{for } 0 \leqslant t \leqslant T.$$

Since $Y_n(t) - X_t \in P_N L^2(\mathscr{C})$ for all $t \in \mathbb{R}_+$, for all n, we get, for given $1 \leqslant p < \infty$,

$$\begin{aligned} \|Y_n(t) - X_t\|_p &= Ce^{sN} \|Y_n(t) - X_t\|_2 \\ &= Ce^{sN} \|w_n - X_T\|_2, \qquad 0 \leqslant t \leqslant T \end{aligned}$$

for suitable constants C and $s \in \mathbb{R}_+$. Q.E.D.

7. Decompositions and Stochastic Integrals in a Probability Gage Space

7.1. We would like to indicate that it is possible to construct an integral of the Itô–Clifford type in a more general context. In what follows we introduce this context and subsequently the construction along with some indication of the motivating case.

We consider an increasing family of finite von Neumann algebras, $(\mathfrak{A}_\alpha)$ $\alpha \in \mathbb{R}^+$, acting on a fixed Hilbert-space $\mathscr{H}$, which satisfy:

(i) If $\alpha_1 \leqslant \alpha_2$ then $\mathfrak{A}_{\alpha_1}$ is a von Neumann subalgebra of $\mathfrak{A}_{\alpha_2}$.

(ii) The algebra $\mathfrak{A}_\infty = (\bigcup_\alpha \mathfrak{A}_\alpha)''$ is finite.

(iii) $\bigcap_{\beta > \alpha} \mathfrak{A}_\beta = \mathfrak{A}_\alpha$.

Let ϕ be a faithful normal trace on $\mathfrak{A}_\infty$ with $\phi(\mathbb{1}) = 1$. We write $L^p(\mathfrak{A}_\alpha, \phi)$, $1 \leqslant p < \infty$, $0 \leqslant \alpha \leqslant \infty$, to denote the non-commutative analogues of the Lebesgue spaces [6, 21, 32]. We shall contract $L^p(\mathfrak{A}_\alpha, \phi)$ to $L^p(\mathfrak{A}_\alpha)$. We note that $L^p(\mathfrak{A}_{\alpha_1}) \subseteq L^p(\mathfrak{A}_{\alpha_2}) \subseteq L^p(\mathfrak{A}_\infty)$ for $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \infty$ and that the conditional expectation $M_\alpha : L^p(\mathfrak{A}_\infty) \to L^p(\mathfrak{A}_\alpha)$ exists and satisfies the properties listed in [29]. A family $(X_\alpha) \subseteq L^1(\mathfrak{A}_\infty)$, $\alpha \in \mathbb{R}^+$ is a *process* if $X_\alpha \in L^1(\mathfrak{A}_\alpha)$. A process is a martingale (resp. supermartingale, submartingale) if $M_{\alpha_1}(X_{\alpha_2}) = X_{\alpha_1}$ for $\alpha_1 \leqslant \alpha_2$ (resp. $M_{\alpha_1}(X_{\alpha_2}) \leqslant X_{\alpha_1}$, $M_{\alpha_1}(X_{\alpha_2}) \geqslant X_{\alpha_1}$). Clearly (X_α) is a supermartingale if and only if $(-X_\alpha)$ is a submartingale. A process (A_α) is positive increasing if $0 \leqslant A_{\alpha_1} \leqslant A_{\alpha_2}$ for $0 \leqslant \alpha_1 \leqslant \alpha_2$. A potential is a supermartingale (X_α) for which $X_\alpha \geqslant 0$, $\alpha \in \mathbb{R}^+$, and $\phi(X_\alpha) \to 0$ as $\alpha \to \infty$. We shall call a process, (X_α), (left, right) continuous if the map: $\mathbb{R}_+ \to L^1(\mathfrak{A}_\infty)$, $\alpha \to X_\alpha$ is (left, right) continuous.

To distinguish between (stochastic) processes which are families of random variables on a probability space and those which are families of (possibly unbounded) operators on Hilbert-space we shall refer to the former as commutative processes and the latter as non-commutative processes. The corresponding theories will be distinguished in the same way. These terms are justified by the fact that integrable random variables on a probability space (Ω, Σ, P) can be identified with elements of the predual of the von Neumann algebra $L^\infty(\Omega, \Sigma, P)$ and conversely if $\mathfrak{A}_\infty$ is commutative then the operators in $L^1(\mathfrak{A}_\infty, \phi)$ may be identified with random variables on an appropriately chosen probability space. The details of this are in [21].

A problem studied extensively in the commutative theory is the Doob–Meyer decomposition of a supermartingale into the difference of a martingale and a positive increasing process. One application of this is the construction of the stochastic integral with respect to an L^2-bounded martingale [15]. We shall develop a non-commutative counterpart to the Doob–Meyer decomposition for this will allow us to define our non-commutative "stochastic" integral in the same way as the usual one. In order for a commutative supermartingale (X_α) to have a Doob–Meyer decomposition it is necessary and sufficient that it be of class D, i.e., that the set of all possible stoppings of (X_α), $\{X_\tau: \tau$ is a stopping time$\}$, be uniformly integrable. Now whilst there is a non-commutative version of the idea of a stopping time and stopped processes (see Sections 3, 6 of [2]) these do not lead to a smooth development of the theory as in the commutative case. However, a result due to Rao [19] indicates a possible way round this. We shall need a little notation. Let (X_α) be a commutative potential. For each $n \in \mathbb{N}$ we have the discrete parameter potential $(X_{i/2^n})_{i=0}^\infty$. Using the Doob decomposition for discrete parameter supermartingales [7] we can write

$$X_{i/2^n} = M_{i/2^n}(A(\infty, n)) - A(i, n),$$

where $A(i, n) = A(i-1, n) - M_{(i-1)/2^n}(X_{i/2^n} - X_{(i-1)/2^n})$ and $A(0, n) = 0$ and $A(\infty, n) = \lim_{i \to \infty} A(i, n)$. We have

7.2 LEMMA. *Let (X_α) be a right continuous potential and let $A(\infty, n)$ be as above. (X_α) is of class D if and only if $\{A(\infty, n): n \in \mathbb{N}\}$ is uniformly integrable.*

This lemma becomes our point of departure in

7.3 DEFINITION. (i) Let $(X_\alpha) \subseteq L^1(\mathfrak{A}_\infty, \phi)$ be a process. We say that (X_α) is of class D if

$$S(X_\alpha) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{n} M_{\alpha_{i-1}}(X_{\alpha_i} - X_{\alpha_{i-1}}): 0 \leqslant \alpha_1 < \alpha_1 \cdots < \alpha_n, n \in \mathbb{N} \right\}$$

is weakly relatively compact in $L^1(\mathfrak{A}_\infty, \phi)$.

(ii) We say that (X_α) is of class D_R if $S_R(X_\alpha) = \{\sum_{i=1}^\infty M_{(i-1)/2^n}(X_{i/2^n} - X_{(i-1)/2^n}): n \in \mathbb{N}\}$ is a weakly convergent sequence in $L^1(\mathfrak{A}_\infty, \phi)$.

Remarks. (i) We replace uniform integrability with weak relative compactness since this is the consequence of uniform integrability we shall actually use.

(ii) This may be a bit stronger than we actually need when dealing with right continuous processes.

(iii) Note that as the weak closure contains the norm closure the objects corresponding to $A(\infty, n)$ will form a weakly relatively compact set.

(iv) Class D_R will correspond to class D in commutative theory. In the commutative case it would be enough to take 7.3(i) as a definition of class D in order to get the decomposition of a class D supermartingale into a martingale and a unique natural positive increasing process. This follows because we can define $\int_0^\infty Y_{S-}\,dA_s$ as a Lebesgue Stieltjes integral (see details of Rao's proof of uniqueness of the natural process). We cannot do this in the non-commutative case so we have to adjust our notion of class D and natural (see below).

(v) In 7.3(ii) the choice of dyadic rationals $i/2^n$ is not essential. All that is required is a sequence $E_n \subseteq \mathbb{R}_+$, where $E_n = \{0 = t_0^n < t_1^n < \cdots < t_m^n \cdots < \infty\}$ and $t_m^n \to \infty$ as $m \to \infty$, with $E_n \subseteq E_{n+1}$ and $\bigcup_{\mathbb{N}} E_n$ a dense subset of $\mathbb{R}^+$.

7.4 EXAMPLE (Ψ is the Clifford Distribution). Let Ψ_α denote $\Psi(u\chi_{[0,\alpha]})$, where $u \in L^2_{\mathbb{R}}(0,\infty)$; we know that Ψ_α is self-adjoint and that $\Psi_\alpha^2 = \|u\chi_{[0,\alpha]}\|_2^2\,\mathbb{1}$. So it is clear that the process (Ψ_α^2) is of class D and D_R since

$$\begin{aligned}\sum_{i=1}^n M_{\alpha_{i-1}}(\Psi^2_{\alpha_i} - \Psi^2_{\alpha_{i-1}}) &= \sum_{i=1}^n M_{\alpha_{i-1}}(\mathbb{1})(\|u\chi_{[0,\alpha_i]}\|_2^2 - \|u\chi_{[0,\alpha]}\|_2^2)\\ &= \sum_{i=1}^n \left(\int_{\alpha_0}^{\alpha_i} |u(s)|^2\,ds - \int_{\alpha_0}^{\alpha_{i-1}} |u(s)|^2\,ds\right)\mathbb{1}\\ &= \left(\int_{\alpha_0}^{\alpha_n} |u(s)|^2\,ds\right)\mathbb{1} \leqslant \|u\|_2^2\,\mathbb{1}.\end{aligned}$$

$\therefore S(\Psi_\alpha^2) \subseteq \{\lambda\mathbb{1} : 0 \leqslant \lambda \leqslant \|u\|_2^2\}$ and so $S(\Psi_\alpha^2)$ is weakly relatively compact.

Before we can exploit this definition we must make one more adaptation. We shall need to consider objects of the form $\int_0^\infty Y_{s-}\,dA_s$ in which $(Y_{s-}) \subseteq \mathfrak{A}_\infty$ is derived from a martingale and (A_s) is a positive increasing process with $A_0 = 0$. In the commutative case this is a Lebesgue Stieltjes integral $\omega \to \int_0^\infty Y_{s-}(\omega)\,dA_s(\omega)$. We shall not (cannot?) define $\int_0^\infty Y_{s-}\,dA_s$ "pathwise" in the non-commutative case but rather we shall define it by using Bartle's bilinear vector integral [4] and by taking a limit. This will allow us to use the argument of Rao's paper [19] which applies to the commutative case. It is probably appropriate to remark at this point that Rao does not specify precisely what is meant by $\int_0^\infty Y_{s-}\,dA_s$. Whilst it is likely that the usual Lebesgue Stieltjes integral is intended the quantities are manipulated very much as if it is a vector integral.

We want to prove

7.5 THEOREM. *Let $(X_\alpha) \subseteq L^1(\mathfrak{A}_\infty)$ be a right continuous supermartingale of class D. Then (X_α) may be decomposed into the difference of a martingale and a positive increasing process. If (X_α) is of class D_R the positive increasing process may be chosen to be natural and in this case the decomposition is unique.*

Remark. We shall not define the term natural at this point since we have yet to make sense of the terms that will define it. We sketch the proof of existence of the decomposition which is just a recasting of the proof of Theorem, Section 1 of [19].

Proof (Existence). Just as in the commutative case one can show that a right continuous supermartingale has a decomposition into the sum of a martingale and a potential (Riesz decomposition). It is enough therefore to consider a right continuous potential of class D. Consider $(X_{i/2^n})_{i=0}^{\infty}$, $i \in \mathbb{N}$; each of these is a discrete parameter supermartingale for which the Doob decomposition is well known,

$$X_{i/2^n} = U(i, n) - A(i, n),$$

where $U(i, n) = U(i-1, n) + X_{i/2^n} - M_{(i-1)/2^n}(X_{i/2^n})$, $U(0, n) = X_0$ and $A(i, n) = A(i-1, n) + X_{(i-1)/2^n} - M_{(i-1)/2^n}(X_{i/2^n})$, $A(0, n) = 0$. Once again we write $A(\infty, n) = L^1 - \lim_i A(i, n)$. Since (X_α) is of class D $\{A(i, n): i, n \in \mathbb{N}\}$ is weakly relatively compact and therefore so is $\{A(\infty, n): n \in \mathbb{N}\}$. This means there is a subsequence converging weakly to some $A \in L^1(\mathfrak{A}_\infty)$ (Eberlein Smulian Theorem) [8]. By writing $U_\alpha = M_\alpha(A)$ and $A_\alpha = U_\alpha - X_\alpha$ we have $X_\alpha = U_\alpha - A_\alpha$ and it can be shown that (A_α) is a positive increasing process with $A_0 = 0$. Using the right continuity of (X_α) and the descending martingale theorem [3] we have that (A_α) is right continuous. Before we consider uniqueness we must give a meaning to $\int_0^\infty f(s)\, dA_s$. Let G denote the ring of sets that are finite disjoint unions of intervals of the form $(s, t]$, $0 \leqslant s \leqslant t < \infty$. Let (A_α) be a positive increasing right continuous process with $A_0 = 0$ and $\sup_\alpha \phi(A_\alpha) < \infty$. We note that $A = \sup_\alpha A_\alpha = \| \ \|_1 - \lim_\alpha A_\alpha$ exists [2]. Define a function on G by $\mu((s, t]) = A_t - A_s$. It is clear that μ is additive and monotone in that if $E, F \in G$ and $E \subseteq F$ then $\mu E \leqslant \mu F$ in the operator sense. The variation of μ,

$$|\mu|\,(E) = \sup \left\{ \sum_{i=1}^{n} \|\mu E_i\|_1 : E_1, \ldots, E_n \in G, \bigcup_{i=1}^{n} E_i = E \right\}, \qquad E \in G,$$

takes on a simple form because μ is positive operator-valued and the L^1-norm is additive on the positive cone, so that

$$\sum_{i=1}^{n} \|\mu E_i\|_1 = \left\| \sum_{i=1}^{n} \mu E_i \right\|_1 = \|\mu E_1\|.$$

Hence $|\mu|$ is the set function $E \mapsto \|\mu E\|_1$ and is monotone and additive. We recall that the semivariation of μ w.r.t. $\mathfrak{A}_\infty$ defined by $\|\mu\|\,(E) = \sup \|\sum_{i=1}^n x_i \mu E_i\|_1$, where the sup is taken over all partitions $(E_i)_{i=1}^n$ of E in G and elements $x_i \in \mathfrak{A}_\infty$ with $\|x_i\|_\infty \leqslant 1$, is dominated by $|\mu|$. However, it is easy to see that in our case $|\mu| = \|\mu\|$ (put $x_i = \mathbb{1}$, $1 \leqslant i \leqslant n$).

Let $\sigma(G)$ denote the σ-field on $(0, \infty)$ generated by G. We shall show that μ has a countably additive extension to $\sigma(G)$.

7.6 Lemma. *μ is countably additive on G.*

Proof. Given an interval $(s, t]$, $0 \leqslant s \leqslant t < \infty$, we can use the right continuity of (A_α) to choose a closed interval $[u, t] \subset (s, t]$ such that, given $\varepsilon > 0$, $|\mu|\,(s, u] < \varepsilon$. Now this allows us to choose for each $\varepsilon > 0$ and $E \in G$ a compact $B \subseteq E$ such that the extension of $|\mu|$ to all subsets of $\mathbb{R}^+$ given by

$$|\mu|\,(A) = \inf\{|\mu|\,(E): E \in G, E \supseteq A\}, \qquad |\mu|\,(\{0\}) = 0$$

(we shall denote this by $|\mu|$ too), is small on $E \backslash B$. Since $|\mu|$ is monotone and subadditive we may argue as follows.

Let (E_n) be a sequence of subsets of $\mathbb{R}^+$ in G decreasing to $\varnothing$. We can find compact $B_n \subseteq E_n$, $|\mu|\,(E_n \backslash B_n) < \varepsilon/2^n$. Since $\bigcap_{\mathbb{N}} B_n = \varnothing$ it follows that there is $r \in \mathbb{N}$ such that $\bigcap_{n=1}^r B_n = \varnothing$. Hence if $m \geqslant r$ then $E_m \subseteq \bigcup_{n=1}^r (E_n \backslash B_n)$. Now $|\mu|$ is monotone; hence,

$$|\mu|\,(E_m) \leqslant |\mu| \left(\sum_{n=1}^{n} (E_n \backslash B_n) \right) \leqslant \sum_{n=1}^{r} |\mu|\,(E_n \backslash B_n) < \varepsilon$$

but $|\mu|\,(\cdot)$ is the map $E \to \|\mu E\|_1$ on G; thus we have shown that $E_n \downarrow \varnothing$ in G implies that $\|\mu E_n\|_1 \downarrow 0$. The monotonicity of μ on G ensures that if (E_n) are disjoint and $\bigcup_{\mathbb{N}} E_n = E \in G$, $E_n \in G$ then $\sum_1^\infty E_n$ exists and $\sum_1^\infty \mu E_n \leqslant \mu E$. With the above then we have $\|\mu E - \sum_{n=1}^m \mu E_n\|_1 \to 0$ as $m \uparrow$. So μ is countably additive on G.

7.7 Corollary. *$|\mu|$ is countably additive on G.*

Proof. $\|\sum_1^\infty \mu E_n\|_1 = \|\mu E\|_1$ if $E_n \in G$, $\bigcup_{\mathbb{N}} E_n = E \in G$, E_n disjoint. But $\|\sum_1^\infty E_n\|_1 = \sum_1^\infty \|\mu E_n\|_1$.

We recall that a vector measure λ satisfies what Bartle calls "the $*$-property" on $\sigma(G)$ if there is a non-negative real-valued measure σ such that $\sigma(E) \to 0 \Leftrightarrow \|\lambda\|\,(E) \to 0$. Bartle also points out that should a measure have finite variation then it satisfies the $*$-property.

Accordingly we have

7.8 Corollary. *μ has a unique countably additive extension to $\sigma(G)$ that satisfies Bartle's $*$-property.*

Proof. That μ has a unique L^1-valued countably additive extension follows from Lemma 8.3.5 of [18] using the fact that $|\mu|$ is countably additive and $\|\mu(E)\|_1 \leqslant |\mu|\,(E)$, $E \in G$. It is an easy calculation to verify that the variation of the extension of μ agrees with the variation of μ. Since

$|\mu|(\mathbb{R}^+) < \infty$ we can conclude that μ satisfies the *-property. (In fact the extension of $|\mu|$ is the positive measure required.) Let the extension of μ be denoted by μ also.

We can now define a countably additive measure on a σ-field of $\mathbb{R}^+$ by treating 0 as a set of measure zero and considering H the σ-field generated by $\sigma(G)$ and $\{0\}$. Bartle's theory shows that there is a space of μ-integrable functions $f: \mathbb{R}^+ \to \mathfrak{A}_\infty$. In the sequel we shall denote $\int_0^\infty f(s)\, d\mu(s)$ by $\int_0^\infty f(s)\, dA_s$.

7.9 Definition. Let (A_α), $\alpha \in \mathbb{R}^+$, be an increasing process with $A_0 \geqslant 0$ and $\sup_\alpha \phi(A_\alpha) < \infty$. Let $Y \in \mathfrak{A}_\infty$ and let $Y_{s-} = \lim_{t \uparrow s} M_t(Y)$, $s \in \mathbb{R}^+$. Define

$$f_n(s) = \sum_{k=1}^{\infty} \chi_{(k-1)/2^n, k/2^n]}(s)\, Y_{(k-1)/2^n}, \qquad f_n(0) = Y_0.$$

For each n, $\int_0^\infty f_n(s)\, dA_s$ exists as a Bartle integral. We say that (A_s) is natural if for each $Y \in \mathfrak{A}_\infty$, $\lim_n \phi(\int_0^\infty f_n(s)\, dA_s) = \phi(AY)$, $A = \sup_\alpha A_\alpha$. By analogy with the commutative case we write this limit as $\phi(\int_0^\infty Y_{s-}\, dA_s)$.

Remark. The existence of Y_{s-} (the limit is taken in $\|\cdot\|_1$) is assured by the martingale theorem.

7.10 Theorem. *Suppose that (X_α) is a potential of class D_R and that (A_α) and (B_α) are two natural processes such that*

$$X_\alpha = M_\alpha(A) - A_\alpha = M_\alpha(B) - B_\alpha, \qquad \alpha \in \mathbb{R}^+;$$

then $A_\alpha = B_\alpha$.

Proof. This is just Rao's proof recast. The point is that $A_\alpha - B_\alpha = M_\alpha(A - B)$; i.e., the processes differ by a martingale; hence

$$\begin{aligned}
\phi\left(\int_0^\infty Y_{s-}\, dA_s\right) &= \lim_n \phi\left(\sum_1^\infty Y_{i/2^n}(A_{(i+1)/2^n} - A_{i/2^n})\right) \\
&= \lim_n \sum_1^\infty \phi(A_{(i+1)/2^n} - A_{i/2^n})) \\
&= \lim_n \sum_1^\infty \phi(Y_{i/2^n} M_{i/2^n}(A_{(i+1)/2^n} - A_{i/2^n})) \\
&= \lim_n \sum_1^\infty \phi(Y_{i/2^n} M_{i/2^n}(B_{(i+1)/2^n} - B_{i/2^n}) \\
&= \phi\left(\int_0^\infty Y_{s-}\, dB_s\right) \qquad \text{by the preceding equations.}
\end{aligned}$$

Since both processes are natural this states that $\phi(YA)=\phi(YB)\ \forall Y\in\mathfrak{A}_\infty^+$, i.e., $A=B$. So $A_\alpha=B_\alpha$.

Proof (Uniqueness). We show that the (A_α) constructed is natural. Once again the proof is just a recasting of that given by Rao. Given $Y\in\mathfrak{A}_\infty$ form $s\mapsto Y_{s-}=\lim_{t\uparrow s}M_t(Y)$. We know that

$$\phi\left(\int_0^\infty Y_{s-}\,dA_s\right)=\lim_n\sum_1^\infty\phi(Y_{i/2^n}(A_{(i+1)/2^n}-A_{i/2^n})).$$

Now $X_\alpha+A_\alpha=U_\alpha\,\forall_\alpha$; hence $M_\alpha(A_{\alpha+s}-A_\alpha)=M_\alpha(X_\alpha-X_{\alpha+s})$ for $s>0$. In particular then $M_{i/2^n}(A_{i+1/2^n}-A_{i//2^n})=M_{i/2^n}(X_{i/2^n}-X_{i+1/2^n})=M_{i/2^n}(M_{i/2^n}(A(\infty,n)-A(i,n))-M_{i/2^n}(A(\infty,n)-A(i+1,n)))=M_{i/2^n}(A(i+1,n)-A(i,n))$, where $(A(i,n))_{i=1}^\infty$ is the increasing process that occurs in the Doob decomposition of the discrete supermartingale $(X_{i/2^n})_{i=1}^\infty$ and $A(\infty,n)=\sup_i A(i,n)$. So

$$\begin{aligned}\phi\left(\int_0^\infty Y_{s-}\,dA_s\right)&=\lim_n\sum_1^\infty\phi(Y_{i/2^n}M_{i/2^n}(A(i+1,n)-A(i,n)))\\&=\lim_n\sum_1^\infty\phi(Y_{i/2^n}A(i+1,n)-Y_{i/2^n}A(i,n))\\&=\lim_n\sum_1^\infty\phi(Y_{i+1/2^n}A(i+1,n)-Y_{i/2^n}A(i,n))\end{aligned}$$

for $Y_{i/2^n}=M_{i/2^n}(Y_{i+1/2^n})$ and $A(i+1,n)\in L^1(\mathfrak{A}_{i/2^n})$. Thus $\phi(\int_0^\infty Y_{s-}\,dA_s)=\lim_n\phi(YA(\infty,n))$ for the last series telescopes, but we know that $\{A(\infty,n): n\in\mathbb{N}\}$ is weakly convergent and that A is the weak limit of this subsequence. We take the limit to get $\phi(\int_0^\infty Y_{s-}\,dA_s)=\phi(YA)$, showing that (A_α) is natural.

7.11 *Remark.* We can now prove the converse of this result, which is a non-commutative version of 7.2 above. Note that the proof of existence and uniqueness of the decomposition proceeds from the weak convergence of $\{A(\infty,n): n\in\mathbb{N}\}$. If (X_α) is a right continuous supermartingale and has a decomposition into a martingale and a positive, natural, increasing process, $X_\alpha=U_\alpha-A_\alpha$, then for a fixed $n\in\mathbb{N}$,

$$A(\infty,n)=\sum_{j=1}^\infty M_{(j-1)/2^n}(X_{(j-1)/2^n}-X_{j/2^n})=\sum_{j=1}^\infty M_{(j-1)/2^n}(A_{j/2^n}-A_{(j-1)/2^n}).$$

So if $Y \in \mathfrak{A}_{\alpha\beta}^{+}$ and we put $Y_{j/2^n} = M_{j/2^n}(Y)$ then

$$\phi(A(\infty, n)\, Y) = \sum_{j=1}^{\infty} \phi((A_{j/2^n} - A_{(j-1)/2^n})\, Y_{(j-1)/2^n})$$

$$= \phi\left(\sum_{j=1}^{\infty} Y_{(j-1)/2^n}(A_{j/2^n} - A_{(j-1)/2^n})\right) = \phi\left(\int_0^{\infty} f_n(s)\, dA_s\right),$$

where $f_n(s) = \sum_{j=1}^{\infty} \chi_{[(j-1)/2^n, j/2^n]}(s) \cdot Y_{(j-1)/2^n}$. Since (A_α) is natural we have

$$\lim_n \phi\left(\int_0^{\infty} f_n\, dA_s\right) = \phi(YA).$$

In the commutative case we would only require $\{A(\infty, n) : n \in \mathbb{N}\}$ weakly relatively compact because we know by Lebesgue's bounded convergence theorem that for each ω outside of a null set

$$\int_0^{\infty} f_n(s, \omega)\, dA_s(\omega) \to \int_0^{\infty} Y_{s-}(\omega)\, dA_s(\omega)$$

so that $\phi(\int_0^{\infty} f_n(s)\, dA_s)$ is convergent. But the weak relative compactness gives a subsequence converging to $\phi(AY)$; hence the *whole* sequence converges to $\phi(AY)$. The problem in the non-commutative context is giving a meaning to $\int_0^{\infty} Y_{s-}\, dA_s$.

7.12. We can now construct an integral with respect to an L^2-martingale. We shall consider the following context. The nest of von Neumann algebras $(\mathfrak{A}_\alpha)$ will be assumed continuous; i.e., we augment (iii) of Section 7.1 by $(\bigcup_{\beta<\alpha} \mathfrak{A}_\beta)'' = \mathfrak{A}_\alpha$. The effect of this is to make all the martingales continuous as maps $\mathbb{R}^+ \to L^1$ rather than just right continuous as above. Let $X = (X_\alpha) \subseteq L^2(\mathfrak{A}_\infty)$ be a bounded martingale. For simplicity suppose that $X_\alpha = X_\alpha^* \forall \alpha$. We know that (X_α^2) is a submartingale. Suppose further that it is of class D. Since $(-X_\alpha^2)$ is an L^1-supermartingale we have $X_\alpha^2 = U_\alpha + A_\alpha$, where (A_α) is a positive natural increasing process and (U_α) is a martingale. Since $\alpha \to X_\alpha$ is $(\|\cdot\|_2)$ continuous, $\alpha \to X_\alpha^2$ is $(\|\cdot\|_1)$ continuous and so $\alpha \to A_\alpha$ is $\|\ \|_1$ continuous. The difference that this makes is that the intervals $(a, b]$, $[a, b)$, $[a, b]$, (a, b) now have the same μ measure.

7.13 Definition. Let $Z = (Z_\alpha)$ be an L^2-martingale. Let $\nu_Z(s, t) = Z_t - Z_s$, $0 \leqslant s \leqslant t < \infty$. This defines an additive function on the ring of sets G. We could prove that ν_X is countably additive on G in the sense that if $(E_n) \subseteq G$, $E_m \cap E_n = \varnothing$ for $m \neq n$ and $\bigcup_n E_n = E \in G$ then $\nu_X(E) =$

$L^2-\lim_k \sum_{n=1}^k \nu_X(E_n)$. This suggests that we could use Bartle's vector integral to define $\int_0^t f(s)\, d\nu_X(s)$ for suitable integrands with values in $L^2(\mathfrak{A}_\infty)$. We shall not do this for reasons which will be clearer once we have constructed the integral in the fashion below. First note that we can make

7.14 DEFINITION. (i) Let $h(s)\colon \mathbb{R}^+ \to \mathfrak{A}_\infty$ be a process. We say h is *simple* if $h(s)=\sum_{i=1}^k h(t_{i-1})\,\chi_{(t_{i-1},t_i]}(x)$ on $(0,\infty)$, where $0 \leqslant t_0 < t_1 < t_2 \cdots < t_k < \infty$.

(ii) Let $h(s)\colon \mathbb{R}^+ \to \mathfrak{A}_\infty$ be a simple process. We define

$$\int_0^t h(s)\, dX_s \stackrel{\text{def}}{=} \int_{[0,t]} h(s)\, d\nu_X = \sum_{i=1}^k h(t_{i-1})\, \nu_X((t_{i-1},t_i] \cap [0,t]).$$

So

$$\begin{aligned} \int_0^t h(s)\, dX_s &= \sum_{i=1}^k h(t_{i-1})(X_{t_i}-X_{t_{i-1}}) \qquad \text{if} \quad t \geqslant t_k \\ &= \sum_{i=1}^j h(t_{j-1})(X_t - X_{t_{j-1}}) \qquad \text{if} \quad t \in (t_{j-1}, t_j]. \end{aligned}$$

7.15 LEMMA. *Let $h(s)$ be a simple $\mathfrak{A}_\infty$-valued process. Then*

$$\left\| \int_0^t h(s)\, dX_s \right\|_2^2 = \phi\left(\int_0^t |h(s)|^2\, dA_s \right),$$

where $dA_s \equiv \mu$ is the measure induced by the natural increasing process in the Doob decomposition of (X_α^2).

Proof. Consider the case $t \geqslant t_k$.

$$\begin{aligned} &\left\| \int_0^t h(s)\, dX_s \right\|_2^2 \\ &= \phi\left(\left(\sum_{i=1}^k h(t_{i-1})(X_{t_i}-X_{t_{i-1}}) \right)^* \left(\sum_{j=1}^k h(t_{j-1})(X_{t_j}-X_{t_{j-1}}) \right) \right) \\ &= \sum_{i,j} \phi((X_{t_i}-X_{t_{i-1}})\, h(t_{i-1})^*\, h(t_{j-1})(X_{t_j}-X_{t_{j-1}})) \qquad \text{by linearity of } \phi, \\ &= \sum_i \phi(|h(t_{i-1})|^2\, (X_{t_i}-X_{t_{i-1}})^2) \end{aligned}$$

because when $i<j$,

$$\begin{aligned}&\phi(M_{t_{j-1}}((X_{t_i}-X_{t_{i-1}})\,h(t_{i-1})^*\,h(t_{j-1})(X_{t_j}-X_{t_{j-1}})))\\&\quad=\phi((X_{t_i}-X_{t_{i-1}})\,h(t_{i-1})^*\,h(t_{j-1})\,M_{t_{j-1}}(X_{t_j}-X_{t_{j-1}}))=0\end{aligned}$$

because $h(s)$ and (X_s) are processes and (X_s) is a martingale. A similar argument shows that the $i>j$ terms vanish. Further, that

$$\begin{aligned}\sum_i \phi(|h(t_{i-1})|^2\,(X_{t_i}-X_{t_{i-1}})^2)&=\sum_i \phi(M_{t_{i-1}}(|h(t_{i-1})|^2\,(X_{t_i}-X_{t_{i-1}})^2))\\&=\sum_i \phi(|h(t_{i-1})|^2\,M_{t_{i-1}}(X_{t_i}-X_{t_{i-1}})^2)\\&=\sum_i \phi(|h(t_{i-1})|^2\,M_{t_{i-1}}(X_{t_i}^2-X_{t_{i-1}}^2))\\&=\sum_i \phi(|h(t_{i-1})|^2\,(A_{t_i}-A_{t_{i-1}}))\\&=\phi\left(\int_0^t |h(s)|^2\,dA_s\right).\end{aligned}$$

The argument for the case $t<t_k$ is identical.

7.16 Definition. Let $\mathscr{P}$ denote the set functions $f\colon \mathbb{R}^+\to\mathfrak{A}_\infty$ such that

(i) f is the μ-almost everywhere limit of simple processes f_n. (i.e., for μ-almost all $s\in[0,\infty)$ we have $\lim_n f_n(s)=f(s)$ in $\|\cdot\|_\infty$.)

(ii) $\operatorname{ess\,sup}_{s<t}\|f(s)\|_\infty<\infty$ for each $t\in\mathbb{R}^+$.

It follows from (i) that if $f\in\mathscr{P}$ then f is a process μ-almost everywhere. From (ii) we see that the sequence may be chosen so that $\forall t>0\ \exists M_t'>0$: $\forall n\ \forall(s<t)\ \|f_n(s)\|_\infty<M_t'$. Theorem 8 of [4] ensures that such an f is μ-integrable and that $L^1-\lim_n \int_0^t f_n(s)\,dA_s=\int_0^t f(s)\,dA_s$.

7.17 Theorem. *Let $f\in\mathscr{P}$ and (f_n) be a sequence of simple processes converging μ-a.e. to f. Then $(\int_0^t f_n(s)\,dX_s)$ converges in $L^2(\mathfrak{A}_\infty)$. The limit, which we denote by $\int_0^t f(s)\,dX_s$, is independent of the particular sequence (f_n) chosen.*

Proof. Consider the case where the sequence of functions (f_n) satisfy $\forall n\ \forall(s<t)\ \|f_n(s)\|_\infty<M$. Given $\delta>0$ let $A_\delta\in H$ be such that $f_n(s)\to f(s)$ uniformly for $s\in[0,t]\backslash A_\delta$ and $\|\mu\|\,(A_\delta)<\delta$ (Egoroff). Choosing $n(\delta)$ so that $\|f_n(s)-f(s)\|_\infty<\delta$ for $n\geqslant n(\delta)$ and $s\in[0,t]\backslash A_\delta$, then if $m,n\geqslant n(\delta)$

$$\left\|\int_0^t f_n\,dX_s - \int_0^t f_m\,dX_s\right\|_2^2 = \phi\left(\int_0^t |f_n - f_m|^2\,dA_s\right)$$

$$= \phi\left(\int_{A_\delta} |f_n - f_m|^2\,dA_s\right) + \phi\left(\int_{[0,t]\backslash A_\delta} |f_n - f_m|^2\,dA_s\right)$$

$$\leqslant \operatorname*{ess\,sup}_{s\in A_\delta} \| |f_n(s) - f_m(s)|^2 \|_\infty \|\mu\|(A)$$

$$+ \operatorname*{ess\,sup}_{s\in[0,t]\backslash A_\delta} \| |f_n(s) - f_m(s)|^2 \|_\infty \|\mu\|([0,t]\backslash A_\delta)$$

$$< 4M^2 \cdot \delta + 4\delta^2 \|\mu\|([0,t]\backslash A_\delta).$$

Now suppose that (f_n) and (g_n) are two sequences of simple processes converging μ-almost everywhere to f. Then $|f_n - g_n|^2$ is a simple process converging to the zero operator μ-almost everywhere We can invoke Theorem 8 of [4] again and conclude that $\int_0^t |f_n - g_n|^2\,dA_s \to \int_0^t 0\,dA_s = 0$ in L^1. So $\phi(\int_0^t |f_n - g_n|^2\,dA_s) = \|\int_0^t f_n\,dX_s - \int_0^t g_n\,dX_s\|_2^2 \to 0$. What we have in fact shown is that $\int_0^t f(s)\,dX_s$ is independent of the sequence of simple processes converging μ-almost everywhere to f.

7.18 Theorem. *Let $f \in \mathscr{P}$; then $(\int_0^t f(s)\,dX_s)$ is a martingale.*

Proof. If $f_n \to f$ μ-a.e. then $\int_0^t f(r)\,dX_r$ is the limit of $\int_0^t f_n(r)\,dX_r$ in $\|\cdot\|_2$. Now let $s < t$; then

$$M_s\left(\int_0^t f(r)\,dX_r\right) = \lim_n M_s\left(\int_0^t f_n(r)\,dX_r\right) = \lim_n \sum_{j=1}^{k(n)} \{f_n(t_{j-1})(X_{t_j} - X_{t_{j-1}})\}$$

for M_s is $\|\cdot\|_2$ continuous and linear. Suppose that $f_n(s)$ has the form $\sum_{j=1}^{k(n)} f_n(t_{j-1})\,\chi_{(t_{j-1},t_j]}(s)$ on $(0,t]$, where $0 = t_0 < t_1 < t_2 \cdots < t_n = t$, and that $s \in (t_{k-1}, t_k]$. Then for $j > k$

$$M_s(f_n(t_{j-1})(X_{t_j} - X_{t_{j-1}})) = M_s M_{t_{j-1}}(f_n(t_{j-1})(X_{t_j} - X_{t_{j-1}})$$
$$= M_s(f(t_{j-1})\,M_{t_{j-1}}(X_{t_j} - X_{t_{j-1}})) = 0,$$

for $j < k$

$$M_s(f_n(t_{j-1})(X_{t_j} - X_{t_{j-1}})) = f_n(t_{j-1})(X_{t_j} - X_{t_{j-1}}),$$

and for $j = k$

$$M_s(f_n(t_{k-1})(X_{t_k} - X_{t_{k-1}})) = f_n(t_{k-1})(X_s - X_{t_{k-1}}).$$

So

$$M_s\left(\int_0^t f_n(x)\,dX_x\right)=\lim_n\int_0^s f_n(x)\,dX_x=\int_0^s f(x)\,dX_x \qquad \text{in} \quad \|\cdot\|_2.$$

Remarks. (i) We can give a meaning to $\int_{t_0}^t f(s)\,dX_s$ in the usual way; it is $\int_0^t f(s)\,\chi_{[t_0,t]}(s)\,dX_s$.

(ii) The class of processes $\mathscr{P}$ is not the widest that can be integrated by (X_s). Part (ii) can be weakened for instance. An open problem is the extension of the integral to $L^2(\mathfrak{A}_\infty)$-valued processes.

(iii) The map $t\to\int_0^t f(s)\,dX_s$, for $f\in\mathscr{P}$, is continuous by the martingale (norm) convergence theorem [3] (conditions 7:1(iii) and 7:10 are essential for this).

(iv) By defining $\mathscr{P}_t=\{f(s)\,\chi_{[0,t]}(s): f\in\mathscr{P}\}$ and defining addition and scalar multiplication pointwise we make $\mathscr{P}_t$ into a vector space. If we identify those elements of $\mathscr{P}_t$ for which $\int_0^t|f-g|^2\,dA_s=0$, and put

$$\langle f,g\rangle=\phi\left(\int_0^t g^*(s)f(s)\,dA_s\right)$$

then $\mathscr{P}_t$ becomes an inner product space and the map $f\in\mathscr{P}_t\xrightarrow{I(f)}\int_0^t f(s)\,dX_s$ is an isometry of $\mathscr{P}_t$ into $L^2(\mathfrak{A}_t)$. We can complete $\mathscr{P}_t$ and define an "abstract" (non-commutative) stochastic integral as the extension of $I(\cdot)$ to the completion of $\mathscr{P}_t$. As remarked in (ii) above the characterisation of the completion has yet to be accomplished for the general case.

(v) The corresponding results to 3:12(a), (b) follow immediately from above.

As remarked above it may be thought that one can define $\int_0^t f(s)\,dX_s$ directly, i.e., by using the measure ν_X and Bartle's vector integral theory. One might also be led to believe that this would lead to a widening of the class of integrands to L^2-valued processes. For this to be so one would require that ν_X had finite semivariation with respect to $L^1(\mathfrak{A}_\infty)$ on bounded intervals of $\mathbb{R}^+$ (because of Theorem 7, 10, and Definition 1, of [4]). However, if this were the case one could choose a continuous $\mathfrak{A}$-valued process, $f(t)$, and it would be true that

$$f_n(t)=\sum_{i=1}^n f\left(\frac{i-1}{n}\right)\chi_{((i-1)/n,\,i/n]}(t)$$

and

$$g_n(t)=\sum_{i=1}^n f\left(\frac{i}{n}\right)\chi_{((i-1)/n,\,i/n]}(t)$$

converged ν_X-almost everywhere to $f(t)$ on $[0, 1]$ and hence that

$$\int_0^1 f(s)\, dX_s = \lim_n \int_0^1 f_n(s)\, dX_s = \lim_n \int_0^1 g_n(s)\, dX_s.$$

If $f(t) = \Psi_t$ and $X_t = \Psi_t$ as in Section 3 above then we could conclude that

$$\left\| \int_0^1 f_n(s)\, dX_s - \int_0^1 g_n(s)\, dX_s \right\|_1 = m\left(\sum_{i=1}^{n} (\Psi_{i/n} - \Psi_{(i-1)/n})^2 \right) = \int_0^1 |u(s)|^2\, ds$$

converges to zero as $n \to \infty$! So ν_X cannot have finite semivariation with respect to $L^2(\mathfrak{A}_\infty)$. This restricts the use of Bartle's theory considerably.

Concluding Remarks. The authors have already begun the task of extending and refining the results above. We are considering the possibility of an Itô–Clifford construction for the type III case using the results in [9]. We have already established some bounded convergence results for the "measure" induced by an L^2-martingale. We hope to present these results soon.

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